

# Global analytic geometry

Frédéric Paugam \*

June 23, 2009

## Abstract

Motivated by the well-known lack of archimedean information in algebraic geometry, we define, formalizing Ostrowski's classification of seminorms on  $\mathbb{Z}$ , a new type of valuation of a ring that combines the notion of Krull valuation with that of a multiplicative seminorm. This definition partially restores the broken symmetry between archimedean and non-archimedean valuations artificially introduced in arithmetic geometry by the theory of schemes. This also allows us to define a notion of global analytic space that reconciles Berkovich's notion of analytic space of a (Banach) ring with Huber's notion of non-archimedean analytic spaces. After defining natural generalized valuation spectra and computing the spectrum of  $\mathbb{Z}$  and  $\mathbb{Z}[X]$ , we define analytic spectra and sheaves of analytic functions on them.

---

\*Université Paris 6, Institut de Mathématiques de Jussieu.

# Contents

<b>1</b>	<b>Halos</b>	<b>5</b>
1.1	Definition and examples . . . . .	5
1.2	Multiplicative elements and localization . . . . .	7
1.3	Tropical halos and idempotent semirings . . . . .	8
1.4	Halos with tempered growth . . . . .	10
1.5	Lexicographic products . . . . .	11
<b>2</b>	<b>Seminorms, valuations and places</b>	<b>12</b>
2.1	Generalizing seminorms and valuations . . . . .	12
2.2	Places . . . . .	16
<b>3</b>	<b>Harmonious spectra</b>	<b>17</b>
3.1	Definition . . . . .	17
3.2	Comparison with Huber's retraction procedure . . . . .	19
3.3	The multiplicative harmonious spectrum of $\mathbb{Z}$ . . . . .	20
3.4	The multiplicative harmonious affine line over $\mathbb{Z}$ . . . . .	22
<b>4</b>	<b>Analytic spaces</b>	<b>27</b>
4.1	Ball topologies and multiplicative completions . . . . .	27
4.2	The functoriality issue for multiplicative completions . . . . .	30
4.3	The analytic spectrum of a ring . . . . .	32
4.4	The analytic spectrum of $\mathbb{Z}$ . . . . .	33
4.5	The analytic affine line over $\mathbb{Z}$ . . . . .	34
<b>5</b>	<b>Another approach to analytic functions</b>	<b>36</b>
5.1	Definition of functorial generalized completions . . . . .	36
5.2	Definition of foanalytic functions . . . . .	37

# Introduction

Many interesting results on polynomial equations can be proved using the mysterious interactions between algebraic, complex analytic and p-adic analytic geometry. The aim of global analytic geometry is to construct a category of spaces which contains these three geometries.

Remark that the study of a given polynomial equation  $P(X, Y) = 0$  is completely equivalent to the study of the corresponding commutative ring  $A = \mathbb{Z}[X, Y]/(P(X, Y))$ . To associate a geometry to a given ring  $A$ , one first needs to define what the points, usually called *places* of this geometry are. There are many different definitions of what a place of a ring is. Kürchák (1912) and Ostrowski (1917) use real valued multiplicative (semi)norms, Krull (1932) uses valuations with values in abstract totally ordered groups and Grothendieck (1958) uses morphisms to fields. There is a natural geometry associated to each type of places:

1. the theory of schemes (see [Gro60]) ensues from Grothendieck's viewpoint,
2. Berkovich's geometry (see [Ber90]) ensues from Ostrowski's viewpoint,
3. Zariski/Huber's geometry (see [Art67] and [Hub93]) ensues from Krull's viewpoint.

For some number theoretical purposes like the study of functional equations of L-functions, a dense part of the mathematical community tend to say that one should try to

“restore the broken symmetry between archimedean and non-archimedean valuations”

artificially introduced in arithmetic geometry by the theory of abstract algebraic varieties (Weil, 1946) and schemes (Grothendieck, 1960), whose great achievements are now patently limited by this symmetry breaking.

As an illustration of this limitation, one can recall that the functional equation

$$\hat{\zeta}(s) = \hat{\zeta}(1 - s)$$

of Riemann's completed zeta function

$$\hat{\zeta}(s) = \pi^{-s/2} \Gamma(s/2) \prod_p \frac{1}{1 - p^{-s}}$$

cannot be studied geometrically without handling the archimedean factor  $\zeta_\infty(s) = \pi^{-s/2} \Gamma(s/2)$  (that corresponds to the archimedean absolute value on  $\mathbb{Q}$ ) in the given geometrical setting. The question is even more interesting for higher dimensional varieties over  $\mathbb{Z}$  because the proof of the functional equation of their zeta function is a widely open question. The theory of schemes will certainly never handle this. Arakelov geometry (see [Sou92] and [Dur07]) partially feels this archimedean gap and this results in a deep improvement of our understanding of the geometry of numbers, but in no proof of the functional equation. A good reason to think that global analytic spaces are useful for this question is the following definition due to Berkovich (private email) which is easily seen to be equivalent to the definition of Tate's thesis [Tat67], which is the corner stone of modern analytic number theory.

**Definition 1.** Let  $|\cdot|_0$  be the trivial seminorm on  $\mathbb{Z}$  and  $U \subset \mathcal{M}(\mathbb{Z})$  be the complement of it in the analytic space of  $\mathbb{Z}$ . Let  $\mathcal{O}$  be the sheaf of analytic functions on  $\mathcal{M}(\mathbb{Z})$ . The adèles of  $\mathbb{Z}$  are the topological ring

$$\mathbb{A} := (j_* \mathcal{O}_U)_{(|\cdot|_0)}$$

of germs of analytic functions at the trivial seminorm.

This geometric definition of adèles opens the road to various higher dimensional generalizations and shows that the topological sheaf of functions on an analytic space is a good replacement of adèles in higher dimensions. It also shows, once combined with the ideas already present in Emil Artin’s book [Art67], that it is worth continuing to think about the following naïve but fundamental question<sup>1</sup>: what is a number?

Another motivation for defining a natural setting for global analytic geometry is that, in the conjectural correspondence between motives and automorphic representations due to Langlands, a lot of (non-algebraic) automorphic representations are left aside. If one enlarges the category of motives by adding the cohomology of natural coefficient systems on analytic varieties, one can hope to obtain a full Langlands correspondence between certain “analytic motivic coefficients” and all automorphic representations. The definition of these analytic motivic coefficients is at this time not at all clear and far beyond the scope of the present paper.

As a first step in the direction of this long term allusive objective, we define in this text a simple notion of generalized valuation (with tempered growth) that allows one to mix the main viewpoints of places in a definition that contains but does not distinguish archimedean and non-archimedean valuations. This definition ensues a new setting of global analytic geometry, that is probably not definitive, but has the merit to give one positive and computable answer to the question: “is it possible to treat all places on equality footing”.

The first construction in the direction of a global analytic geometry is due to Berkovich [Ber90], chapter 1 (see also Poineau’s thesis [Poi07]): he considers spaces of multiplicative seminorms on commutative Banach rings, giving the example of the Banach ring  $(\mathbb{Z}, |\cdot|_\infty)$  of integers with their archimedean norm. He defines a category of global analytic spaces that contains complex analytic and his non-archimedean analytic spaces. One of the limitations of his construction is that a good theory of non-archimedean coherent analytic sheaves sometimes imposes the introduction of a Grothendieck topology (the rigid analytic topology defined by Tate [Tat71]) on his analytic spaces, which is essentially generated by affinoid domains  $\{x \mid |a(x)| \leq |b(x)| \neq 0\}$ . It was proved by Huber [Hub93] that in the non-archimedean case, the topos of sheaves for this Grothendieck topology has enough points, so that it corresponds to a usual topological space. This space is the valuation spectrum of the corresponding adic ring, whose points are bounded continuous Krull valuations. The non-archimedean components of Berkovich’s analytic spaces give subspaces (or more precisely retractions) of Huber’s valuation spectra corresponding to rank one valuations. However, there is no construction in the literature that combines

---

<sup>1</sup>Question which will not get a satisfying answer in this paper.

Huber's viewpoint (which is nicer from an abstract sheaf theoretic point of view) with Berkovich's viewpoint (which has the advantage of giving separated spaces and allowing to naturally incorporate archimedean components).

We propose in this text a new kind of analytic spaces that gives a natural answer to this simple open problem. The construction is made in several steps. We start in Section 1 by studying the category of halos, which is the simplest category that contains the category of rings, and such that Krull valuations and seminorms are morphisms in it. In Section 2, we define a new notion of tempered generalized valuation which entails a new notion of place of a ring. In Section 3, we use this new notion of place to define a topological space called the harmonious spectrum of a ring. In Section 4, we give a definition of the analytic spectrum and define analytic spaces using local model similar to Berkovich's [Ber90], 1.5. We finish by computing in detail the points of the global analytic affine line over  $\mathbb{Z}$  and proposing another approach to the definition of analytic functions.

All rings and semirings of this paper will be unitary, commutative and associative.

# 1 Halos

We want to define a category that contains rings fully faithfully and such that valuations and (multiplicative) seminorms both are morphisms in this category. The most simple way to do this is to use the category whose objects are semirings equipped with a partial order compatible to their operations and whose morphisms are maps  $f : A \rightarrow B$  such that  $f(0) = 0$ ,  $f(1) = 1$ , and that fulfill the subadditivity and submultiplicativity conditions

$$\begin{aligned} f(a + b) &\leq f(a) + f(b), \\ f(ab) &\leq f(a)f(b). \end{aligned}$$

An object of this category will be called a halo. It is often supposed, for localization purposes, that  $f$  is strictly multiplicative, i.e.,  $f(ab) = f(a)f(b)$ . We will see that this hypothesis is sometimes too restrictive for our purposes.

## 1.1 Definition and examples

**Definition 2.** A halo is a semiring  $A$  whose underlying set is equipped with a partial order  $\leq$  which is compatible to its operations:  $x \leq z$  and  $y \leq t$  implies  $xy \leq zt$  and  $x + y \leq z + t$ . A morphism between two halos is an increasing map  $f : A \rightarrow B$  which is submultiplicative, i.e.,

- $f(1) = 1$ ,
- $f(ab) \leq f(a)f(b)$  for all  $a, b \in A$ ,

and subadditive, i.e.,

- $f(0) = 0$ ,

- $f(a + b) \leq f(a) + f(b)$  for all  $a, b \in A$ .

The category of halos is denoted  $\mathfrak{H}\mathfrak{alos}$ . A halo morphism is called square-multiplicative (resp. power-multiplicative, resp. multiplicative) if  $f(a^2) = f(a)^2$  (resp.  $f(a^n) = f(a)^n$ , resp.  $f(ab) = f(a).f(b)$ ) for all  $a, b \in A$  and  $n \in \mathbb{N}$ . The categories of halos with square-multiplicative (resp. power-multiplicative, resp. multiplicative) morphisms between them is denoted  $\mathfrak{H}\mathfrak{alos}^{sm}$  (resp.  $\mathfrak{H}\mathfrak{alos}^{pm}$ , resp.  $\mathfrak{H}\mathfrak{alos}^m$ ).

Let  $B$  be a semiring. The trivial order on  $B$  gives it a halo structure that we will denote  $B_{triv}$ . If  $A$  is a halo and  $f : A \rightarrow B_{triv}$  is a halo morphism, then  $f$  is automatically a semiring morphism. The functor  $B \mapsto B_{triv}$  gives a fully faithful embedding of the category of semirings into the categories  $\mathfrak{H}\mathfrak{alos}$ ,  $\mathfrak{H}\mathfrak{alos}^{sm}$ ,  $\mathfrak{H}\mathfrak{alos}^{pm}$  and  $\mathfrak{H}\mathfrak{alos}^m$ .

*Remark 1.* The field  $\mathbb{R}$  equipped with its usual order is not a halo because this order is not compatible with the multiplication of negative elements. This shows that a halo is something different of the usual notion of an ordered ring used in the literature.

We will now prove that rings have only one halo structure: the trivial one.

**Lemma 1.** *A halo which is a ring has necessarily a trivial order.*

*Proof.* It is mainly the existence of an inverse for addition which implies that the order is trivial. Suppose that  $a \leq b \in A$ . Since  $-b - a = -b - a$  and the sum respects the order, we have  $-b - a + a \leq -b - a + b$ , i.e.  $-b \leq -a$ . We know that  $-1 = -1$  so  $(-1).(-b) \leq (-1).(-a)$ . Now adding  $b$  to  $0 = (-1 + 1).(-b) = (-1).(-b) + (-b)$  implies  $(-1).(-b) = b$ . So we have  $b \leq a$ , which implies that  $b = a$ .  $\square$

*Remark 2.* From now on, we will often identify a ring with its unique (trivial) halo structure.

**Definition 3.** A halo whose underlying semiring is a semifield is called an aura.

**Definition 4.** A halo  $A$  is called positive if  $0 < 1$  in  $A$ .

*Remark 3.* If a halo  $A$  is positive, then  $0 = 0.a \leq 1.a = a$  for all  $a \in A$ .

*Remark 4.* If a totally ordered aura  $R$  is positive and  $0 < r \neq 1$  in  $R$ , then there exists  $0 < r' < r$  in  $R$ . Indeed, if  $r > 1$ , then  $r' = 1/r < 1 < r$  and if  $r < 1$ , then  $r' = r^2 < r$ .

We now give three interesting examples of halo morphisms.

*Example 1.* The semifield  $\mathbb{R}_+$  equipped with its usual laws and ordering is a totally ordered positive aura. If  $A$  is a ring, then a classical seminorm on  $A$  is exactly a halo morphism

$$|\cdot| : A \rightarrow \mathbb{R}_+.$$

*Example 2.* More generally, if  $R$  is a real closed field, the semifield  $R_{\geq 0} = \{x^2 | x \in R\}$  of its positive elements (i.e. its positive cone, that is also its squares since  $R$  is real closed) equipped with its usual laws and ordering is a totally ordered positive aura. If  $A$  is a ring, we can thus generalize seminorms by using halo morphisms

$$|\cdot| : A \rightarrow R_{\geq 0}.$$

These have the advantage to be tractable with model theoretic methods because the theory of real closed fields admits elimination of quantifiers (see [Sch90]).

*Example 3.* If  $\Gamma$  is a totally ordered group (multiplicative notation), the semigroup  $R_\Gamma := \{0\} \cup \Gamma$  equipped with the multiplication and order such that  $0 \cdot \gamma = 0$  and  $0 \leq \gamma$  for all  $\gamma \in \Gamma$  and with addition  $a + b = \max(a, b)$  is also a totally ordered positive aura. Its main difference with the positive cone of a real closed field is that it does not have the simplification property

$$x + y \leq x + z \Rightarrow y \leq z.$$

If  $A$  is a ring, a halo morphism

$$|\cdot| : A \rightarrow R_\Gamma$$

is exactly a valuation in Krull's sense. If  $\Gamma^{div}$  is the divisible closure of  $\Gamma$  and  $R = \mathbb{R}((\Gamma^{div}))$  is the corresponding real closed field, we can associate to  $|\cdot|$  a halo morphism

$$|\cdot| : A \rightarrow R_{\geq 0}$$

given by composition with the natural multiplicative halo morphism  $R_\Gamma \rightarrow R_{\geq 0}$ .

## 1.2 Multiplicative elements and localization

**Definition 5.** Let  $f : A \rightarrow B$  be a halo morphism. The set of multiplicative elements for  $f$  in  $A$  is the set  $M_f$  of  $a \in A$  such that

1.  $f(a) \in B^\times$ ,
2. for all  $b \in A$ ,  $f(ab) = f(a)f(b)$ .

**Proposition 1.** Let  $A$  be a halo and  $S \subset A$  be a multiplicative subset (i.e. a subset that contains 1 and is stable by multiplication). Then the localized semiring  $A_S$  is equipped with a natural halo structure such that if  $f : A \rightarrow B$  is a halo morphism with  $S \subset M_f$  then  $f$  factorizes uniquely through the morphism  $A \rightarrow A_S$ .

*Proof.* The localized semiring  $A_S$  is defined as the quotient of the product  $A \times S$  by the relation

$$(a, s)R(b, t) \Leftrightarrow \exists u \in S, uta = usb.$$

Recall that the sum and product on  $A \times S$  are defined by  $(a, s) + (b, t) = (at + bs, st)$  and  $(a, s) \cdot (b, t) = (ab, st)$ . We put on  $A \times S$  the pre-order given by

$$(a_1, s_1) \leq (a_2, s_2) \Leftrightarrow \exists u \in S, ua_1s_2 \leq us_1a_2.$$

Remark that the equivalence relation associated to this pre-order is exactly the equivalence relation we want to quotient by. This order is compatible with the two operations given above. Indeed, if  $(a_1, s_1) \leq (a_2, s_2)$  and  $(b_1, t_1) \leq (b_2, t_2)$ , then by definition there exist  $u, v \in S$  with  $ua_1s_2 \leq us_1a_2$  and  $vb_1t_2 \leq vt_1b_2$ , so that  $uva_1b_1s_2t_2 \leq uva_2b_2s_1t_1$  by compatibility of the order with multiplication in  $A$ . This shows that  $(a_1, s_1) \cdot (b_1, t_1) \leq (a_2, s_2) \cdot (b_2, t_2)$ . Now  $(a_1, s_1) + (b_1, t_1) = (a_1t_1 + b_1s_1, s_1t_1)$  and  $(a_2, s_2) + (b_2, t_2) = (a_2t_2 + b_2s_2, s_2t_2)$ . Remark that  $(a_1t_1 + b_1s_1) \cdot s_2t_2 = a_1t_1s_2t_2 + b_1s_1s_2t_2$  and  $(a_2t_2 + b_2s_2) \cdot s_1t_1 = a_2t_2s_1t_1 + b_2s_2s_1t_1$ . The inequalities  $ua_1s_2 \leq us_1a_2$  and  $vb_1t_2 \leq vt_1b_2$  imply  $ua_1s_2 \cdot (t_1t_2v) \leq$

$us_1a_2.(t_1t_2v)$  and  $vb_1t_2.(s_1s_2u) \leq vt_1b_2.(s_1s_2u)$ . Adding these inequalities and changing parenthesis, we get  $(uv).(a_1t_1 + b_1s_1).(s_2t_2) \leq (uv).(a_2t_2 + b_2s_2).(s_1t_1)$  which shows that  $(a_1, s_1) + (b_1, t_1) \leq (a_2, s_2) + (b_2, t_2)$ , as claimed. So the operations on  $A \times S$  are compatible with the defined pre-order. The quotient order of this pre-order is exactly the underlying set of the localized semiring, and is equipped with a canonical order compatible to its operations, i.e. a canonical halo structure. Let  $f : A \rightarrow B$  be a halo morphism such that  $S \subset M_f$ . Then we can define  $\tilde{f} : A \times S \rightarrow B$  by  $\tilde{f}(a, s) = f(a)/f(s)$ . This is well defined since  $f(s)$  is invertible in  $B$ . Suppose now that  $(a_1, s_1) \leq (a_2, s_2)$  in  $A \times S$ , which means that there exists  $u \in S$  such that  $ua_1s_2 \leq us_1a_2$ . We then have  $f(ua_1s_2) \leq f(us_1a_2)$  and since  $S \subset M_f$ , this gives  $f(a_1)f(s_2) \leq f(s_1)f(a_2)$  so that  $\tilde{f}(a_1, s_1) \leq \tilde{f}(a_2, s_2)$ . This shows that  $\tilde{f}$  factorizes through  $A_S$ . Now it remains to show that the obtained map, denoted  $g$ , is also subadditive and submultiplicative. We already know that for all  $a, b \in A$ ,  $f(a + b) \leq f(a) + f(b)$  and  $f(ab) \leq f(a).f(b)$ . Remark that by definition of  $g$  and since  $S \subset M_f$ , we have

$$g\left(\frac{a}{s} \cdot \frac{b}{t}\right) = \frac{f(ab)}{f(st)} = \frac{f(ab)}{f(s)f(t)} \leq \frac{f(a)}{f(s)} \cdot \frac{f(b)}{f(t)} = g(a/s) \cdot g(b/t).$$

We also have  $g(a/s + b/t) = g(\frac{at+bs}{st}) = g(at + bs)/g(st)$  and

$$g(at + bs)/g(st) \leq g(at)/g(st) + g(bs)/g(st) = g(a)/g(s) + g(b)/g(t).$$

This shows that  $g$  is a halo morphism.  $\square$

**Corollary 1.** *Let  $A$  be a halo and  $S \subset A$  be a multiplicative subset. The localized semiring  $A_S$  is equipped with a natural halo structure such that if  $f : A \rightarrow B$  is a multiplicative halo morphism with  $f(S) \subset B^\times$  then  $f$  factorizes uniquely through the morphism  $A \rightarrow A_S$ .*

### 1.3 Tropical halos and idempotent semirings

**Definition 6.** A halo  $A$  is called tropical if it is non-trivial, totally ordered and  $a + b = \max(a, b)$  for all  $a, b \in A$ .

If  $A$  is a positive totally ordered halo, we denote  $A_{trop}$  the same multiplicative monoid equipped with its tropical addition  $a +_{trop} b := \max(a, b)$ . There is a natural halo morphism  $A_{trop} \rightarrow A$ . Remark that tropical halos usually don't have the simplification property

$$x + y \leq x + z \Rightarrow y \leq z.$$

**Definition 7.** Let  $A$  be a positive halo.  $A$  is called archimedean if  $A$  is not reduced to  $\{0, 1\}$  and for all  $x > y > 0$ , there exists  $n \in \mathbb{N}$  with  $ny > x$ . Otherwise,  $A$  is called non-archimedean.

**Lemma 2.** *A tropical halo is positive and non-archimedean.*

*Proof.* Let  $A$  be a tropical halo. This implies that  $0 \neq 1$  because  $A$  is non-trivial. Suppose that  $1 \leq 0$  in  $A$ . Then  $1 + 0 := \max(1, 0) = 0 \neq 1$ , which is a contradiction with the fact that  $A$  is a semiring. If  $A$  is reduced to  $\{0, 1\}$ , then it is non-archimedean. Now suppose that  $A$  is not reduced to  $\{0, 1\}$ . If  $a > b > 0$  then  $a > n.b = b$  for all  $n \in \mathbb{N}$  so that  $A$  is non-archimedean.  $\square$



We will now show that tropical halos and idempotent semirings are related.

**Definition 8.** A semiring  $A$  is called idempotent if  $a + a = a$  for all  $a \in A$ .

Let  $A$  be an idempotent semiring. The relation

$$a \leq b \Leftrightarrow a + b = b$$

is a partial order relation on  $A$  that gives  $A$  a halo structure denoted  $A_{halo}$ . This will be called the natural halo structure of the idempotent semiring.

*Example 4.* The semiring  $\mathbb{R}_{+,trop}$  of positive real numbers with usual multiplication and tropical addition given by  $a +_{trop} b = \max(a, b)$  is an idempotent semiring which is tropical. The semiring  $\mathbb{R}_{+,trop}[X]$  of polynomials is idempotent but not tropical because  $X$  and  $1$  cannot be compared in it:  $X + 1 \neq X$  and  $X + 1 \neq 1$ .

**Definition 9.** Let  $K$  be a tropical halo and  $S$  be a set. Then the polynomial semiring  $K[S]$  is idempotent and is thus equipped with a natural halo structure. This halo will be called the halo of polynomials on  $K$ .

The following Lemma shows that the order of a tropical halo is of a purely algebraic nature.

**Lemma 3.** *The functor  $A \mapsto A_{halo}$  induces an equivalence of categories between idempotent semirings whose natural order is total and tropical halos with multiplicative morphisms.*

*Proof.* First remark that if  $f : A \rightarrow B$  is a semiring morphism between two idempotent semirings, then  $a \leq b$  in  $A$  implies  $a + b = b$  so that  $f(a) + f(b) = f(b)$  and  $f(a) \leq f(b)$  in  $B$ , which means that  $f$  is an increasing map. This shows that the map  $A \mapsto A_{halo}$  is a functor. If  $A$  is a tropical halo, then its underlying semiring is idempotent. The natural order of this semiring is equal to the given order and this last one is total. This shows that the functor  $A \mapsto A_{halo}$  is essentially surjective from the category of idempotent semirings with total natural order to tropical halos. Let  $A$  and  $B$  be two tropical halos. A halo morphism  $f : A \rightarrow B$  is increasing so that  $f(\max(a, b)) = \max(f(a), f(b))$  and  $f$  is a semiring morphism. This shows that the functor  $A \mapsto A_{halo}$  is full. It is also faithful because  $A$  and  $A_{halo}$  have the same underlying set.  $\square$

*Remark 5.* Usual semirings with their trivial order and tropical halos are two subcategories of the category  $\mathfrak{H}alos^m$  of halos with multiplicative morphisms that share a common feature: all their morphisms are strictly additive, i.e., semiring morphisms.

**Definition 10.** Let  $\Gamma$  be a multiplicative totally ordered monoid and  $R_\Gamma := \{0\} \cup \Gamma$ . We equip  $R_\Gamma$  with a tropical halo structure by declaring that  $0$  is smaller than every element of  $\Gamma$ , annihilates every element of  $\Gamma$  by multiplication, and that  $a + b = \max(a, b)$  for all  $a, b \in R_\Gamma$ . The halo  $R_\Gamma$  is called the tropical halo of  $\Gamma$ . If  $\Gamma$  is a group, the tropical halo  $R_\Gamma$  is an aura.

*Example 5.* Let  $\Gamma$  be a totally ordered group and  $H \subset \Gamma$  be a convex subgroup (i.e.  $g < h < k$  in  $\Gamma$  and  $g, k \in H$  implies  $h \in H$ ). Then  $\pi_H : R_\Gamma \rightarrow R_{\Gamma/H}$  is a surjective halo morphism between tropical auras.

*Example 6.* Let  $R_{\{1\}} = \{0, 1\}$  be the tropical halo on the trivial group. It is equipped with the order given by  $0 \leq 1$ , idempotent addition given by  $1 + 1 = 1$  and usual multiplication. It is the initial object in the category of positive halos (in which  $0 \leq 1$ ), so in particular in the category of tropical halos. Indeed, if  $A \neq 0$  is such a halo, then the injective map  $f : R_{\{1\}} \rightarrow A$  that sends 0 to 0 and 1 to 1 is a halo morphism because  $0 \leq 1$  implies  $f(1 + 1) = f(1) = 1 \leq 1 + 1 = f(1) + f(1)$ .

**Definition 11.** A halo is called trivial if it is reduced to  $\{0\}$  or equal to the tropical halo  $R_{\{1\}}$ .

## 1.4 Halos with tempered growth

If  $A$  and  $R$  are two halos, halo morphisms  $|\cdot| : A \rightarrow R$  are not easy to compute in general. We now introduce a condition that can be imposed on  $R$  to make this computation easier. This condition is directly inspired by Ostrowski's classification of multiplicative seminorms on  $\mathbb{Z}$ .

**Definition 12.** A halo  $R$  has tempered growth if for all non-zero polynomial  $P \in \mathbb{N}[X]$ ,

$$x^n \leq P(n) \text{ in } R \text{ for all } n \in \mathbb{N} \text{ implies } x \leq 1.$$

**Lemma 4.** A tropical halo has tempered growth.

*Proof.* Let  $R$  be a tropical halo. Then  $n = 1 +_{trop} \cdots +_{trop} 1 = 1$  in  $R$  for all  $n \in \mathbb{N}$  so that  $P(n) = 1$  in  $R$  for all  $n \in \mathbb{N}$  and  $P \in \mathbb{N}[X]$  non-zero. Let  $P$  be such a polynomial. Suppose that  $a^n \leq P(n)$  for all  $n \in \mathbb{N}$ . In particular, we have  $a \leq P(1) = 1$  which shows that  $R$  has tempered growth.  $\square$

**Lemma 5.** Let  $R$  be a non-trivial totally ordered positive aura in which  $x > y$  implies that there exists  $t > 0$  such that  $x = y + t$ . Suppose moreover that  $\mathbb{N}$  injects in the underlying semiring of  $R$ . If  $R$  is archimedean then  $R$  has tempered growth.

*Proof.* Let  $R$  be as in the hypothesis of this Lemma. Let  $P \in \mathbb{N}[X]$  be a non-zero polynomial of degree  $d$  and suppose there exists  $x > 1$  such that  $x^n \leq P(n)$  in  $R$  for all  $n \neq 0$ . By hypothesis, we can write  $x = 1 + t$  with  $t > 0$  and  $x^n = (1 + t)^n = 1 + nt + \frac{n(n-1)}{2}t^2 + \cdots$ . The components of this sum are all positive so that  $\frac{n!}{p!(n-p)!}t^p \leq P(n)$ . Write  $P(n) = \sum a_i n^i$ . Since  $R$  is archimedean, we can choose  $n = n.1 > \max(a_i)$ . Now since  $R$  is a totally ordered aura,  $n^i \leq n^j$  for all  $i \leq j$  so that  $P(n) = \sum a_i n^i \leq (d+1)n^{d+1}$ . We have proved that  $\frac{n!}{p!(n-p)!}t^p \leq (d+1)n^{d+1}$  in  $R$  for all  $n$  big enough in  $\mathbb{N}$ . If we take  $n > 2p$  and  $p = d+1$ , we get

$$n(n-1)(n-2)\cdots(n-d)t^p = \frac{n!}{(n-p)!}t^p \leq (d+1)p!n^{d+1},$$

where the left product has  $d + 2$  terms. Remark now that  $n - i \geq n - (d + 1)$  implies  $\frac{n}{n-i} \leq \frac{n}{n-(d+1)}$  in  $R$ . We also have  $n > 2(d + 1)$  implies  $n - (d + 1) \geq \frac{n}{2}$  in  $R$ , so that  $\frac{n}{n-i} \leq 2$  for  $i \leq d + 1$ , which shows that

$$(n - p)t^p \leq (d + 1)p!2^{d+1}.$$

Since  $t > 0$  and  $R$  is a totally ordered aura,  $t^p > 0$ . Moreover, the equality  $t^p = (d + 1)p!2^{d+1}$  for all successive  $p$  is not possible. Indeed, this would give  $t = \frac{t^{p+1}}{t^p} = \frac{(p+1)!}{p!} = p+1$  and  $t = p + 2$  for a convenient  $p$  so that  $p + 1 = p + 2$  which is a contradiction with the fact that  $\mathbb{N}$  injects in  $R$ . We thus have  $t^p < (d + 1)p!2^{d+1}$  and since  $R$  is archimedean, we know that there exists  $n$  big enough such that  $(n - p)t^p > (d + 1)p!2^{d+1}$ . This gives a contradiction.  $\square$

**Corollary 2.** *The aura  $\mathbb{R}_+$  has tempered growth.*

*Proof.* This follows directly from Lemma 5. We can also give a more direct proof using a little bit of real analysis. Let  $x \in \mathbb{R}_+$  and  $P \in \mathbb{N}[X]$  be such that  $x^n \leq P(n)$  for all  $n \in \mathbb{N}$ . Then taking  $n$ -th root and passing to the limit, we get

$$x \leq \lim_{n \rightarrow \infty} \sqrt[n]{P(n)} = 1.$$

$\square$

We will see in the next Section some nice examples of archimedean auras in which  $\mathbb{N}$  embeds but that have non-tempered growth, showing that the hypothesis of Lemma 5 are optimal.

*Remark 6.* We know from Lemma 2 that a tropical halo  $A$  is non-archimedean. This shows that being of tempered growth is not equivalent to being archimedean.

## 1.5 Lexicographic products

Let  $R_1, \dots, R_n$  be a finite family of positive auras. Equip  $\prod R_i$  with its lexicographic order. Remark that  $\prod R_i^\times \subset \prod R_i$  is a multiplicative submonoid that is stable by addition because  $a, b > 0$  in  $R_i$  implies  $a + b > 0$ . We extend this embedding to  $\{0\} \cup \prod R_i^\times$  sending 0 to  $(0, \dots, 0)$ . We will denote  $[\prod] R_i := \{0\} \cup \prod R_i^\times$  with its halo structure induced by its embedding into  $\prod R_i$ . This halo is automatically an aura.

**Definition 13.** Let  $R_1, \dots, R_n$  be a finite list (i.e. ordered family) of positive auras. The aura  $[\prod] R_i$  is called the lexicographic product of the family. If  $R$  is a positive aura, we denote  $R^{[n]}$  the lexicographic product  $\left[\prod_{1, \dots, n}\right] R$ . If  $R$  and  $S$  are positive aura, we denote  $R_{[\times]} S$  their lexicographic product.

Remark that if the  $R_i$  are totally ordered, then so is  $[\prod] R_i$ .

**Lemma 6.** *If  $n > 1$ , the aura  $\mathbb{R}_+^{[n]}$  is archimedean but it does not have tempered growth.*

*Proof.* Suppose that  $0 < (x_i) < (y_i)$  in  $\mathbb{R}_+^{[n]}$ . Then at least  $0 < x_1 \leq y_1$  in  $\mathbb{R}_+$  so that there exists  $n \in \mathbb{N}$  such that  $nx_1 > y_1$ , which implies  $n(x_i) > (y_i)$  in  $\mathbb{R}_+^{[n]}$ . This shows that  $\mathbb{R}_+^{[n]}$  is archimedean. Now let  $a > 1$  in  $\mathbb{R}$ . Then  $(1, \dots, 1, a) > (1, \dots, 1)$  in  $\mathbb{R}_+^{[n]}$  but since  $1 < n + 2$ ,  $(1, \dots, 1, a)^n = (1, \dots, 1, a^n) < (n + 2, \dots, n + 2) = n + 2 \in \mathbb{R}_+^{[n]}$ . This shows that  $\mathbb{R}_+^{[n]}$  does not have tempered growth.  $\square$

**Lemma 7.** *Let  $K$  be a tropical aura and  $R$  be an aura that has tempered growth. The aura  $K_{[\times]}R$  has tempered growth.*

*Proof.* Let  $P \in \mathbb{N}[X]$  be a non-zero polynomial and  $(x, y) \in K_{[\times]}R$  be such that  $(x, y) > 1 = (1, 1)$  and  $(x, y)^n \leq (P(n), P(n))$  for all  $n$ . Remark that  $P(n) = 1 \in K$  for all  $n$ .  $(x, y)^1 \leq (P(1), P(1)) = (1, P(1))$  implies  $x \leq 1$ . If  $x < 1$ , then  $(x, y) < (1, 1)$  which is a contradiction. If  $x = 1$  then  $y > 1$ . Remark that  $(x, y)^n = (1, y^n) \leq (P(n), P(n)) = (1, P(n))$  for all  $n$  implies  $y^n \leq P(n)$  for all  $n$ . Since  $R$  has tempered growth, this means that  $y \leq 1$  in  $R$ , which is a contradiction. We thus have proved that  $K_{[\times]}R$  has tempered growth.  $\square$

**Corollary 3.** *The aura  $\mathbb{R}_{+,trop}[\times]\mathbb{R}_+$  has tempered growth.*

## 2 Seminorms, valuations and places

Some proofs of the forthcoming Sections are very similar to their classical version, which one can find in E. Artin's book [Art67] and in Bourbaki [Bou64], *Algèbre Commutative*, Chap. VI.

### 2.1 Generalizing seminorms and valuations

**Definition 14.** A generalized seminorm on a ring  $A$  is a halo morphism from  $A$  to a positive totally ordered aura  $R$ , i.e., a map  $|\cdot| : A \rightarrow R$  from  $A$  to a positive totally ordered semifield  $R$  such that

1.  $|1| = 1, |0| = 0$ ,
2. for all  $a, b \in A$ ,  $|ab| \leq |a| \cdot |b|$ ,
3. for all  $a, b \in A$ ,  $|a + b| \leq |a| + |b|$ .

A generalized seminorm  $|\cdot| : A \rightarrow R$  is called

- square-multiplicative if  $|a^2| = |a|^2$ ,
- power-multiplicative if  $|a^n| = |a|^n$  for all  $a \in A$  and all  $n \in \mathbb{N}$ ,
- tempered if  $R$  has tempered growth,
- non-archimedean if

$$|a + b| \leq \max(|a|, |b|)$$

for all  $a, b \in A$ ,

- pre-archimedean if

$$|a + b| \leq \max(|2|, 1) \cdot \max(|a|, |b|)$$

for all  $a, b \in A$ .

We will often omit “generalized” in “generalized seminorm”.

*Remark 7.* Let  $A$  be a ring. A generalized seminorm on  $A$  with values in  $\mathbb{R}_+$  is exactly a seminorm on  $A$  in the usual sense. A multiplicative generalized seminorm on  $A$  with value in a tropical aura  $R_\Gamma = \{0\} \cup \Gamma$  is exactly a valuation in Krull’s sense (multiplicative notation).

Let  $A$  be a ring and  $|\cdot| : A \rightarrow R$  be a generalized seminorm on  $A$ . Then  $\text{Ker}(|\cdot|)$  is an ideal in  $A$ . Indeed, if  $|a| = 0$  and  $|b| = 0$ , then  $|a + b| \leq |a| + |b| = 0$  so that  $|a + b| = 0$ . If  $|a| = 0$  and  $b \in A$ , then  $|a \cdot b| \leq |a| \cdot |b| = 0$ . Remark also that  $|\cdot| : A \rightarrow R$  factorizes through  $A/\text{Ker}(|\cdot|)$ . Indeed, if  $|a| = 0$  and  $b \in A$ , then  $|b| = |b + a - a| \leq |b + a| + |-a| \leq |b + a| + |-1| \cdot |a| = |b + a|$  and  $|b + a| \leq |b| + |a| = |b|$ , which shows that  $|b + a| = |b|$ . If  $|\cdot|$  is multiplicative (resp. square-multiplicative, resp. power-multiplicative) then its kernel is a prime (resp. square-reduced, resp. reduced) ideal.

**Lemma 8.** *A seminorm  $|\cdot|$  always fulfills  $|-1| \geq 1$ . If it is square-multiplicative, it moreover fulfills  $|-a| = |a|$  for all  $a \in A$ .*

*Proof.* If  $|-1| < 1$  then  $1 = |1| = |(-1)^2| \leq |-1|^2 \leq 1 \cdot |-1| = |-1|$  which is a contradiction. Suppose now that  $|\cdot|$  is square-multiplicative. We then have  $|-1| = 1$ . Indeed, if  $|-1| > 1$  then  $1 = |1| = |(-1)^2| = |-1|^2 \geq |-1| \cdot 1 = |-1|$  which is a contradiction. We also have  $|-a| = |a|$  for all  $a \in A$ . Indeed, we have  $|-a| \leq |-1| \cdot |a| = |a|$  and  $|a| = | -(-a) | \leq |-a|$  so that  $|-a| = |a|$ .  $\square$

*Example 7.* The map  $|\cdot| = |\cdot|_{6,0} : \mathbb{Z} \rightarrow R_{\{1\}} := \{0, 1\}$  given by setting  $|n| = 0$  if  $6|n$  and 1 otherwise is not multiplicative because  $|2 \cdot 3| = 0 < |2| \cdot |3| = 1$  but it is power-multiplicative. Similarly, the 6-adic seminorm  $|\cdot| = |\cdot|_6 : \mathbb{Z} \rightarrow \mathbb{R}_+$  that sends a number  $n$  to  $6^{-\text{ord}_6(n)}$  is power-multiplicative but not multiplicative because  $|2 \cdot 3|_6 = 1/6 < |2|_6 \cdot |3|_6 = 1$ .

*Remark 8.* If we stick to  $\mathbb{R}_+$ -valued seminorms, one can show that power-multiplicative seminorms correspond (through the supremum construction) to compact subsets of the space of bounded  $\mathbb{R}_+$ -valued multiplicative seminorms on the corresponding completion, as explained by Berkovich in [Ber90], chapter 1. Moreover, the power-multiplicativity condition can be shown to be equivalent to the square-multiplicativity condition  $|a^2| = |a|^2$  for all element  $a$  of the algebra (by using the spectral radius). The advantage of the square-multiplicative formulation is that it is defined by a first order logic condition that is easier to deal with using model theoretic methods. Next Lemma shows that the notion of pre-archimedean seminorm (which is also adapted to model theoretic method) allows an easier determination of non-archimedean valuations.

**Lemma 9.** *Let  $|\cdot| : A \rightarrow R$  be a pre-archimedean seminorm, i.e.,*

$$|a + b| \leq \max(|2|, 1) \cdot \max(|a|, |b|)$$

*for all  $a, b \in A$ . Then the following conditions are equivalent:*

1.  $|\cdot|$  is non-archimedean,
2.  $|n| \leq 1$  for all integer  $n \in \mathbb{N}$ ,
3.  $|2| \leq 1$ .

Moreover, if  $R_2^* \subset R^*$  is the convex subgroup generated by  $|2|$ , then the induced map  $|\cdot|' : A \rightarrow R_{R^*/R_2}$  is a non-archimedean seminorm.

*Proof.* The first condition implies the second because of the ultrametric inequality. The second implies the third. If  $|2| \leq 1$  and  $|\cdot|$  is pre-archimedean then  $|\cdot|$  is non-archimedean because  $\max(|2|, 1) = 1$ . This shows the equivalence. Because of the pre-archimedean condition, and since  $|2| = |1|$  in  $R_{R^*/R_2}$ , we conclude that  $|\cdot|'$  is non-archimedean.  $\square$

We now define two notions of equivalence of halo morphisms.

**Definition 15.** Let  $A$  be a halo, let  $|\cdot|_1 : A \rightarrow R$  and  $|\cdot|_2 : A \rightarrow S$  be two halo morphisms from  $A$  to two given halos. We say the  $|\cdot|_1$  is bounded (resp. multiplicatively bounded) by  $|\cdot|_2$  and we write  $|\cdot|_1 \leq |\cdot|_2$  (resp.  $|\cdot|_1 \leq_m |\cdot|_2$ ) if for all  $a, b \in A$  (resp. for all  $a, b, c \in A$ ),

$$\begin{aligned} &|a|_2 \leq |b|_2 \Rightarrow |a|_1 \leq |b|_1 \\ \text{(resp. } &|a|_2 \cdot |c|_2 \leq |b|_2 \Rightarrow |a|_1 \cdot |c|_1 \leq |b|_1 \text{).} \end{aligned}$$

We say that  $|\cdot|_1$  is equivalent (resp. multiplicatively equivalent) to  $|\cdot|_2$  if

$$\begin{aligned} &|\cdot|_1 \leq |\cdot|_2 \text{ and } |\cdot|_2 \leq |\cdot|_1 \\ \text{(resp. } &|\cdot|_1 \leq_m |\cdot|_2 \text{ and } |\cdot|_2 \leq_m |\cdot|_1 \text{).} \end{aligned}$$

Remark that multiplicative equivalence is stronger than equivalence and allows one to transfer multiplicativity properties between equivalent seminorms. In particular, if two seminorms  $|\cdot|_1 : A \rightarrow R_1$  and  $|\cdot|_2 : A \rightarrow R_2$  are multiplicatively equivalent, then their multiplicative subsets in  $A$  are equal. If we denote it  $M$ , by Proposition 1, we thus get factorizations  $|\cdot|_1 : A_M \rightarrow R_1$  and  $|\cdot|_2 : A_M \rightarrow R_2$  that remain multiplicatively equivalent.

The proof of the following theorem, that was our main motivation to introduce the notion of tempered seminorm, is a refinements of Artin's proof of Ostrowski's classification of absolute values on  $\mathbb{Z}$ .

**Theorem 1.** Let  $|\cdot| : A \rightarrow R$  be a tempered power-multiplicative seminorm on a ring  $A$ . Then  $|\cdot|$  is pre-archimedean, i.e. fulfills

$$|a + b| \leq \max(|2|, 1) \cdot \max(|a|, |b|)$$

for all  $a, b \in A$ . If moreover  $|2| > 1$  then  $|\cdot|_{\mathbb{Z}}$  is multiplicatively equivalent to the archimedean seminorm  $|\cdot|_{\infty} : \mathbb{Z} \rightarrow \mathbb{Q}_+$  given by  $|n|_{\infty} = \max(n, -n)$ .

*Proof.* We can first suppose that  $A = \mathbb{Z}$ . Let  $n, m > 1$  be two natural numbers. We may write  $m^s = a_0 + a_1 n + a_r n^{r(s)}$  where  $a_i \in \{0, 1, \dots, n-1\}$  and  $n^{r(s)} \leq m^s$ . More precisely,  $r(s)$  is the integral part  $\left[ s \cdot \frac{\log m}{\log n} \right]$  of  $s \cdot \frac{\log m}{\log n}$ , so that there exists a constant  $c_{m,n} \in \mathbb{N}$  such

that  $r(s) \leq s.c_{m,n}$  for all  $s \in \mathbb{N}$ . In fact, we can choose  $c_{m,n} = 1 + \left\lceil \frac{\log m}{\log n} \right\rceil$ . Now remark that  $|a_i| = |1 + \dots + 1| \leq a_i \cdot |1| \leq n$  for all  $i$ . We now use that  $|\cdot|$  is power-multiplicative, so that

$$|m|^s = |m^s| \leq \sum_{i=0}^{r(s)} |a_i| \cdot |n|^i \leq \sum |a_i| \max(1, |n|)^{r(s)} \leq n(1 + r(s)) \max(1, |n|)^{r(s)}.$$

Since  $r(s) \leq s.c_{m,n}$ , we get finally

$$|m|^s \leq n(1 + s.c_{m,n}) \max(1, |n|)^{s.c_{m,n}}.$$

This gives

$$\left( \frac{|m|}{\max(1, |n|)^{c_{m,n}}} \right)^s \leq n(1 + s.c_{m,n}) = P(s),$$

for  $P(X) = n(1 + X.c_{m,n}) \in \mathbb{N}[X]$ . Since  $R$  has tempered growth, this implies that  $\left( \frac{|m|}{\max(1, |n|)^{c_{m,n}}} \right) \leq 1$ , i.e.,

$$|m| \leq \max(1, |n|)^{c_{m,n}}$$

for  $c_{m,n} \in \mathbb{N}$ .

First suppose that  $|2| \leq 1$ . Then, applying the above inequality with  $n = 2$ , we get  $|m| \leq 1$  for all  $m \in \mathbb{Z}$ . If  $a, b \in A$ , we have

$$|a + b|^s = |(a + b)^s| \leq \sum_{i=0}^s \left| \binom{s}{i} \right| \cdot \max(|a|, |b|)^s \leq (s + 1) \max(|a|, |b|)^s$$

because  $\binom{s}{i}$  is an integer so that  $|\binom{s}{i}| \leq 1$ . Since  $R$  has tempered growth, this inequality implies

$$|a + b| \leq \max(|a|, |b|).$$

Now suppose that  $1 < |2|$ . Then applying the above inequality with  $m = 2$ , we get  $1 < |2| \leq \max(1, |n|)^{c_{m,n}}$  so that  $1 < \max(1, |n|)$  and  $|n| > 1$  for all non-zero  $n \in \mathbb{Z}$ .

Now suppose that  $m > n > 1$ . Then  $\log(m) > \log(n)$  so that  $\frac{\log m}{\log n} > 1$  and  $c_{m,n} = 1 + \left\lceil \frac{\log m}{\log n} \right\rceil \geq 2$ . We also have  $c_{n,m} \leq 1$ , which implies that  $c_{n,m} = 1$ , so that  $|n| \leq |m|^{c_{n,m}} = |m|$ . We thus have proved that  $|\cdot| : \mathbb{N} \rightarrow R$  is an increasing map for the usual order on  $\mathbb{N}$ .

As before, if  $a, b \in A$ , we have

$$|a + b|^s = |(a + b)^s| \leq \sum_{i=0}^s \left| \binom{s}{i} \right| \cdot \max(|a|, |b|)^s.$$

We recall for reader's convenience that, since the binomial coefficient  $\binom{s}{i}$  counts the number of parts of cardinal  $i$  in a set of cardinal  $s$  and  $2^s$  is the cardinal of the set of parts of a set of cardinal  $s$ , we have  $\binom{s}{i} \leq 2^s$ . Since  $|\cdot| : \mathbb{N} \rightarrow R$  is increasing, we get  $|\binom{s}{i}| \leq |2^s|$  and

$$|a + b|^s \leq (s + 1)(|2| \cdot \max(|a|, |b|))^s,$$

and since  $R$  has tempered growth, this inequality implies

$$|a + b| \leq |2| \cdot \max(|a|, |b|).$$

Together with what we showed at the beginning, this implies that  $|\cdot|$  is pre-archimedean.

It remains to prove that it is injective. Since  $\frac{\log n}{\log m} < 1$ , there exists a rational number  $p/q$  such that  $\frac{\log n}{\log m} < p/q < 1$ . Now if we work in the usual real numbers, we have the inequality  $n = m^{\frac{\log n}{\log m}} \leq m^{p/q}$  so that  $n^q \leq m^p$  in the ordered set  $\mathbb{N}$  of integers  $\mathbb{N}$ . Since  $|\cdot|$  is increasing, we get  $|n|^q \leq |m|^p$  with  $p < q$ . Changing  $p/q$  to  $(up)/(uq)$  with  $u > 1$ , we can suppose that  $q - p > 1$ . Now suppose that  $|m| = |n|$ . This implies  $|n|^q \leq |n|^p$  with  $p < q$  and since  $|n| > 1$ , this gives  $|n|^{q-p} \leq |n|$  with  $q - p > 1$ . But since  $|n| > 1$ , this gives a contradiction.

Now suppose that for  $m > n > 1$ , we have  $0 < |mn| < |m| \cdot |n|$ . This implies that for all  $q \in \mathbb{N} - \{0\}$ , we have

$$1 < \left( \frac{|m| \cdot |n|}{|mn|} \right)^q.$$

Remark that we have  $mn = m^{\frac{\log n}{\log m} + 1}$ , so that for all rational number  $p/q$  such that  $\frac{\log n}{\log m} < p/q < 1$ , we have  $n \leq m^{p/q}$  so that  $n^q \leq m^p$  and  $|m|^q \cdot |n|^q \leq |n|^{p+q}$ . We also have  $p < q$  so that  $(mn)^q \geq (mn)^p = m^p n^p \geq n^{p+q}$  and

$$1 < \left( \frac{|m| \cdot |n|}{|mn|} \right)^q \leq \frac{|n|^{p+q}}{|n^{p+q}|} \leq 1$$

(by power-multiplicativity) which is a contradiction. We thus get that  $|m| \cdot |n| \leq |mn|$  and  $|\cdot|_{\mathbb{Z}}$  is multiplicative and multiplicatively equivalent to  $|\cdot|_{\infty}$ .  $\square$

## 2.2 Places

**Definition 16.** A place of a ring is a multiplicative equivalence class of generalized seminorm. If  $x$  is a place of a ring  $A$ , we denote  $|\cdot(x)| : A \rightarrow R$  a given representative of  $x$ .

If the representative  $|\cdot(x)|$  of a given place  $x$  of  $A$  is multiplicative, then all other representatives, being multiplicatively equivalent to it, will also be multiplicative. This shows that our notion of place of a ring generalizes the classical notion.

The  $p$ -adic valuation  $|\cdot|_{p, \text{trop}} : \mathbb{Z} \rightarrow R_{p^{\mathbb{Z}}}$  and the  $p$ -adic seminorm  $|\cdot|_p : \mathbb{Z} \rightarrow \mathbb{R}_+$  are equivalent tempered multiplicative seminorms. They thus represent the same place of  $\mathbb{Z}$ .

The use of non-multiplicative seminorms in analytic geometry is imposed by the central role played by the notion of uniform convergence on compacts in the theory of complex analytic functions. For example, if  $K \subset \mathbb{C}$  is a compact subset, we want the seminorm

$$|\cdot|_{\infty, K} : \mathbb{C}[X] \rightarrow \mathbb{R}_+$$

given by  $|P|_{\infty, K} := \sup_{x \in K} |P(x)|_{\mathbb{C}}$  with  $|\cdot|_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{R}_+$  the usual complex norm to be a (square-multiplicative) place of  $\mathbb{C}[X]$ .



### 3 Harmonious spectra

We now want to define a notion of spectrum of a ring that combines the valuation (or Zariski-Riemann) spectrum with the seminorm spectrum. It will be called the harmonious spectrum. We define various versions of this space that are adapted to the various problem we want to solve.

#### 3.1 Definition

**Definition 17.** Let  $A$  be a ring. We define various spaces of seminorms on  $A$ .

1. The multiplicative harmonious spectrum of  $A$  is the set  $\text{Speh}^m(A)$  of multiplicative tempered places on  $A$ .
2. The power-multiplicative harmonious spectrum of  $A$  is the set  $\text{Speh}^{pm}(A)$  of tempered power-multiplicative places on  $A$ .
3. The pre-archimedean square-multiplicative harmonious spectrum of  $A$  is the set  $\text{Speh}^{pasm}(A)$  of pre-archimedean square-multiplicative places of  $A$ .
4. The Krull valuation spectrum of  $A$  is the set  $\text{Spev}(A) = \text{Speh}^v(A)$  of equivalence classes of Krull valuations, i.e., multiplicative tropical seminorms  $|\cdot| : A \rightarrow R_\Gamma$ .

For  $\bullet \in \{v, m, pm, pasm\}$ , the topology on  $\text{Speh}^\bullet(A)$  is generated by subsets of the form

$$U\left(\frac{a}{b}\right) = \{x \in \text{Speh}^\bullet(A) \mid |a(x)| < |b(x)|, d|b \Rightarrow d \text{ is multiplicative for } |\cdot(x)|\}.$$

The spaces  $\text{Speh}^{pasm}(A)$  and  $\text{Spev}(A)$  can be studied by model theoretic methods as the ones used in [Pre98] and [HK94] because they are defined in the setting of first order logic.

*Remark 9.* There are also good reasons to use the topology generated by subsets of the form

$$U\left(\frac{a}{b}\right) = \{x \in \text{Speh}^\bullet(A) \mid |a(x)| \leq |b(x)| \neq 0, d|b \Rightarrow d \text{ is multiplicative for } |\cdot(x)|\}.$$

These are better compatible with Huber's version of analytic spaces and don't seem to be so incompatible with archimedean analytic geometry as was explained to us by Huber (private mail).

*Remark 10.* The topology on  $\text{Speh}^m(A)$  is generated by subsets of the form  $U\left(\frac{a}{b}\right) = \{x \in \text{Speh}^\bullet(A) \mid |a(x)| < |b(x)|\}$  because the multiplicativity condition on  $b$  is automatic. We can also see that in any case, the topology on  $\text{Speh}^\bullet(A)$  has rational domains of the form

$$R\left(\frac{a_1, \dots, a_n}{b}\right) = \{x \in \text{Speh}^\bullet(A) \mid |a_i(x)| < |b(x)|, d|b \Rightarrow d \text{ is multiplicative for } |\cdot(x)|\}$$

as a basis. Indeed, the intersection of two rational domains  $R\left(\frac{f_1, \dots, f_n}{h}\right)$  and  $R\left(\frac{g_1, \dots, g_m}{k}\right)$  is the rational domain  $R\left(\frac{f_1 k, \dots, f_n k, g_1 h, \dots, g_m h}{hk}\right)$ .

Let  $\mathcal{M}(A)$  denote the set of multiplicative  $\mathbb{R}_+$ -valued seminorms, equipped with the coarsest topology that makes the maps  $x \mapsto |a(x)|$  for  $a \in A$  continuous.

We recall the following for reader's convenience (see [Poi08] for details on spectrally convex subsets).

**Definition 18.** Let  $V$  be a compact subset of  $\mathcal{M}(A)$ . Denote  $\|\cdot\|_{\infty, V}$  the supremum of all seminorms in  $V$  and  $\mathcal{B}(V)$  the completion of the localization of  $A$  by the set of elements of  $A$  that are non-zero in a neighborhood of  $V$ . We call  $V$  spectrally convex if the natural map

$$\mathcal{M}(\mathcal{B}(V)) \rightarrow \mathcal{M}(A)$$

has image  $V$ .

**Proposition 2.** *There is a natural bijection between the set of  $\mathbb{R}_+$ -valued square-multiplicative seminorms on  $A$  and the set of spectrally convex compact subsets of  $\mathcal{M}(A)$ .*

*Proof.* The fact that power-multiplicative  $\mathbb{R}_+$ -valued seminorms correspond to compact subsets of the topological space of bounded multiplicative seminorms on a Banach algebra is explained in [Ber90], Section 1.2. To a given real-valued power-multiplicative seminorm  $|\cdot| : A \rightarrow \mathbb{R}_+$ , one can associate the compact subset of  $\mathcal{M}(A)$  given by all multiplicative seminorms  $|\cdot|' : A \rightarrow \mathbb{R}_+$  bounded by it, i.e., such that  $|\cdot|' \leq |\cdot|$ . By using the spectral radius, one shows that every square-multiplicative  $\mathbb{R}_+$ -valued seminorm is automatically power-multiplicative.  $\square$

By applying Theorem 1 and Proposition 2, we get a map from the set of compact subsets of  $\mathcal{M}(A)$  to  $\text{Speh}^{pasm}(A)$  that gives a bijection between spectrally convex subsets of  $\mathcal{M}(A)$  and points in  $\text{Speh}^{pasm}(A)$  given by  $\mathbb{R}_+$ -valued seminorms.

**Lemma 10.** *The Krull valuation spectrum  $\text{Spev}(A)$  embeds in  $\text{Speh}^m(A)$ , and can be defined in this space as the subspace*

$$\text{Spev}(A) = \{x \in \text{Speh}^m(A) \mid |2(x)| \leq 1\} = \{x \in \text{Speh}^m(A) \mid \forall n \in \mathbb{N}, |n(x)| \leq 1\}.$$

*Proof.* First remark that we know by Theorem 1 that all seminorms in  $\text{Speh}^m(A)$  are pre-archimedean, and by Lemma 9, the hypothesis implies that they are non-archimedean, i.e., fulfill that for all  $a, b \in A$ ,

$$|a + b| \leq \max(|a|, |b|)$$

if and only if  $|2| \leq 1$ , and this is also equivalent to  $|n| \leq 1$  for all  $n \in \mathbb{N}$ . The subset described above is exactly  $\text{Spev}(A)$ .  $\square$

We have the following natural diagram of continuous maps.

$$\begin{array}{ccccc} & & \text{Spev}(A) & \longrightarrow & \text{Speh}^{pm}(A) \\ & & \downarrow & & \downarrow \\ \mathcal{M}(A) & \longrightarrow & \text{Speh}^m(A) & \longrightarrow & \text{Speh}^{pasm}(A) \end{array}$$

*Remark 11.* The spaces  $\text{Speh}^m(A)$  and  $\text{Speh}^{pm}(A)$  are defined by quantifying on integers (because of the temperation and power-multiplicativity hypothesis). The spaces  $\text{Speh}^{pasm}(A)$  and  $\text{Spev}(A)$  have the advantage of being defined in the setting of first order logic. This makes them quite well adapted to model theoretic methods. Unfortunately, we are not able to use them directly to define analytic spaces.

### 3.2 Comparison with Huber's retraction procedure

Roland Huber explained to us the following results, which are of great interest for our study. He also kindly authorized us to include his original ideas in our article, ideas which are completely from his own mind, and are localized in this subsection.

If  $x = |\cdot(x)| : A \rightarrow R$  is a multiplicative seminorm on a ring  $A$ , we denote  $\Gamma_x$  the totally ordered group  $|K(x)| - \{0\}$  of non-zero elements in the image of the extension of  $x$  to the fraction field  $K(x)$  of  $A/\text{Ker}(|\cdot(x)|)$  where  $\text{Ker}(|\cdot(x)|)$  is the prime ideal of element of  $A$  whose image by  $|\cdot(x)|$  is 0.

Huber considers the space  $\text{Speh}^{pam}(A)$  of pre-archimedean multiplicative seminorms (called separated quasi-valuations by him) on a ring  $A$  equipped with the topology generated by the sets  $\{x \in \text{Speh}^{pam}(A) \mid |a(x)| \leq |b(x)| \neq 0\}$ ,  $a, b \in A$ . This space is defined in the setting of first order logic and can thus be studied with model theoretic tools. It is spectral and the boolean algebra of constructible subsets is generated by the above subsets.

The subsets of seminorms  $x$  such that  $|2(x)| > 1$  (resp.  $|2(x)| \leq 1$ ) is called the archimedean (resp. non-archimedean) subset of  $\text{Speh}^{pam}(A)$ . It is a closed (resp. open) constructible subset of  $\text{Speh}^{pam}(A)$  denoted by  $\text{Speh}_a^{pam}(A)$  (resp.  $\text{Speh}_{na}^{pam}(A)$ ), and there is an equality of topological spaces

$$\text{Speh}_{na}^{pam}(A) = \text{Speh}^v(A) = \text{Spev}(A).$$

For the purpose of composing seminorms, Huber defines a subset  $\text{Speh}_{sep}^{pam}(A)$  of  $\text{Speh}^{pam}(A)$  composed of separated seminorms. These are seminorms  $|\cdot(x)| : A \rightarrow R$  such that for every  $\gamma \in \Gamma_x$  with  $\gamma > 1$ , there is some  $n \in \mathbb{N}$  with  $\gamma^n > |2|$  (in particular, every non-archimedean seminorm in there is quasi-separated).

Huber remarked that the natural map

$$\text{Speh}^m(A) \rightarrow \text{Speh}_{sep}^{pam}(A)$$

from tempered multiplicative seminorms (in the sense of our article) to pre-archimedean multiplicative and separated seminorms is a bijection. Moreover, we have an equality of topological spaces

$$\text{Speh}_{sep,na}^{pam}(A) = \text{Speh}_{na}^{pam}(A) = \text{Spev}(A)$$

as said above.

For every  $x \in \text{Speh}^{pam}(A)$ , put

$$\Delta_x = \{\gamma \in \Gamma_x \mid (\max(1, |2(x)|))^{-1} \leq \gamma^n \leq \max(1, |2(x)|), \text{ for every } n \in \mathbb{N}\}.$$

Then  $\Delta_x$  is a convex subgroup of  $\Gamma_x$ . If  $x$  is non-archimedean then  $\Delta_x = \{1\}$ . If  $x$  is archimedean, then  $\Delta_x$  is the greatest convex subgroup of  $\Gamma_x$  that does not contain  $|2(x)|$ . We have a retraction from  $\text{Speh}^{pam}(A)$  onto its subset  $\text{Speh}_{sep}^{pam}(A)$ ,

$$r : \text{Speh}^{pam}(A) \rightarrow \text{Speh}_{sep}^{pam}(A)$$

given by  $x \mapsto x/\Delta_x$ . The quotient topology  $\tau_{sep}$  of the given topology on  $\text{Speh}^{pam}(A)$ , called the retraction topology on  $\text{Speh}_{sep}^{pam}(A)$ , gives a very nice combination of the closed inequalities topology on the non-archimedean part with the open inequalities topology on the archimedean part.

To be more precise, let us describe the open subset of  $\tau_{sep}$ . For every  $q \in \mathbb{Q}$ , define a function  $\lambda_q$  on  $\text{Speh}^{pam}(A)$  by setting  $\lambda_q(x) = (\max(1, |2(x)|))^q \in \Gamma_x \otimes \mathbb{Q}$ . For all  $a, b \in A$ , the set  $D(a, b)_q := \{x \in \text{Speh}^{pam}(A) \mid |a(x)| \leq \lambda_q(x) \cdot |b(x)| \neq 0\}$  is open and constructible in  $\text{Speh}^{pam}(A)$ . Put

$$\begin{aligned} U(a, b)_q &:= \bigcup_{t \in \mathbb{Q}, t < q} D(a, b)_t \\ U_{sep}(a, b)_q &:= U(a, b)_q \cap \text{Speh}_{sep}^{pam}(A). \end{aligned}$$

Then  $U(a, b)_q$  is open in  $\text{Speh}^{pam}(A)$  and  $U(a, b)_q = r^{-1}(U_{sep}(a, b)_q)$ . Hence  $U_{sep}(a, b)_q$  is open in  $\tau_{sep}$ . The non-archimedean part of  $U_{sep}(a, b)_q$  is equal to  $\{x \in \text{Spev}(A) \mid |a(x)| \leq |b(x)| \neq 0\}$ , i.e., a standard non-archimedean open subset. The archimedean part of  $U_{sep}(a, b)_q$  is equal to  $\{x \in \text{Speh}_{sep, a}^{pam}(A) \mid |a(x)| < \lambda_q(x) \cdot |b(x)|\}$ .

Huber further shows that the topology on the archimedean component  $\text{Speh}_{sep, a}^{pam}(A)$  is generated by the sets  $U_{sep}(a, b)_0$  and that  $\text{Speh}_{sep, na}^{pam}(A)$  is a dense open subset of  $\text{Speh}_{sep}^{pam}(A)$ . This implies that the topology on the non-archimedean (resp. archimedean) part of  $\text{Speh}_{sep}^{pam}(A)$  is generated by subsets of the form

$$\begin{aligned} &\{x \mid |a(x)| \leq |b(x)| \neq 0\} \\ &(\text{resp. } \{x \mid |a(x)| < |b(x)|\}), \end{aligned}$$

for  $a, b \in A$ , so that the retraction topology combines nicely the open inequalities topology of archimedean analytic geometry with the closed inequalities topology of non-archimedean analytic geometry.

This shows that the space  $\text{Speh}^{pam}(A)$  and its relations to global analytic geometry deserve to be further studied.

### 3.3 The multiplicative harmonious spectrum of $\mathbb{Z}$

We will now show that the harmonious spectrum of  $\mathbb{Z}$  is very close to the previously known spectrum of  $\mathbb{Z}$ .

**Lemma 11.** *Let  $K$  be a finite field and  $|\cdot| : K \rightarrow R$  be a multiplicative seminorm. Then  $|\cdot|$  is multiplicatively equivalent to the trivial seminorm  $|\cdot|_0 : K \rightarrow R_{\{1\}} = \{0, 1\}$ .*

*Proof.* If  $n$  is the order of  $K^\times$ , then  $x^n = 1$  for all  $x \in K^\times$ . We can suppose  $n > 1$ . Let  $x \in K^\times$  and suppose that  $|x| \neq 1$ . We always have  $|x|^n = 1$ . If  $|x| < 1$ , then  $|x|^{n-1} \leq 1$  so that  $|x|^n \leq |x|$ . But  $|x|^n = 1$ , which implies that  $1 \leq |x| < 1$ . This is a contradiction. If we suppose  $|x| > 1$ , we also arrive to  $1 \geq |x| > 1$ . This shows that  $|x| = 1$ .  $\square$

**Lemma 12.** *Let  $|\cdot| : \mathbb{Q} \rightarrow R$  be a non-archimedean multiplicative seminorm on  $\mathbb{Q}$  with trivial kernel. Then  $|\cdot|$  is either equivalent to  $|\cdot|_p : \mathbb{Q} \rightarrow K_{p^{\mathbb{Z}}}$  for a prime number  $p$  or to  $|\cdot|_0 : \mathbb{Q} \rightarrow K_{\{1\}} = \{0, 1\}$ .*

*Proof.* We have  $|n| = |1 + \dots + 1| \leq 1$ . If  $|p| = 1$  for all primes, then  $|n| = 1$  for all  $n$  because of unique factorization. This implies that  $|\cdot|$  is equivalent to  $|\cdot|_0$ . Suppose now that there exists a prime  $p$  such that  $|p| < 1$ . The set  $\mathfrak{p} = \{a \in \mathbb{Z} \mid |a| < 1\}$  is an ideal of  $\mathbb{Z}$  such that  $p\mathbb{Z} \subset \mathfrak{p} \neq \mathbb{Z}$ . Since  $p\mathbb{Z}$  is a maximal ideal, we have  $\mathfrak{p} = p\mathbb{Z}$ . If now  $a \in \mathbb{Z}$  and  $a = bp^m$  with  $b$  not divisible by  $p$ , so that  $b \notin \mathfrak{p}$ , then  $|b| = 1$  and hence  $|a| = |p|^m$ . Remark now that  $|p| < 1$  implies  $|p|^{n+1} < |p|^n$  for all  $n$  so that the map  $|\cdot| : \mathbb{Q} \rightarrow R$  factorizes through the tropical field  $K_{|p|^{\mathbb{Z}}} = 1 \cup \{|p|^{\mathbb{Z}}\} \subset R$ . We have thus proved that  $|\cdot|$  is equivalent to  $|\cdot|_p$ .  $\square$

**Proposition 3.** *There is a natural identification*

$$\text{Speh}^m(\mathbb{Z}) = \{|\cdot|_p, |\cdot|_{p,0}, p \text{ prime}\} \cup \{|\cdot|_\infty\}$$

*of the multiplicative tempered spectrum of  $\mathbb{Z}$  with the set consisting of  $p$ -adic and  $p$ -residual seminorms for all prime ideals  $(p)$  (including  $(p) = (0)$ ), and of the archimedean norm. The natural map  $\mathcal{M}(\mathbb{Z}) \rightarrow \text{Speh}^m(\mathbb{Z})$  is surjective and can be described by figure 1.*

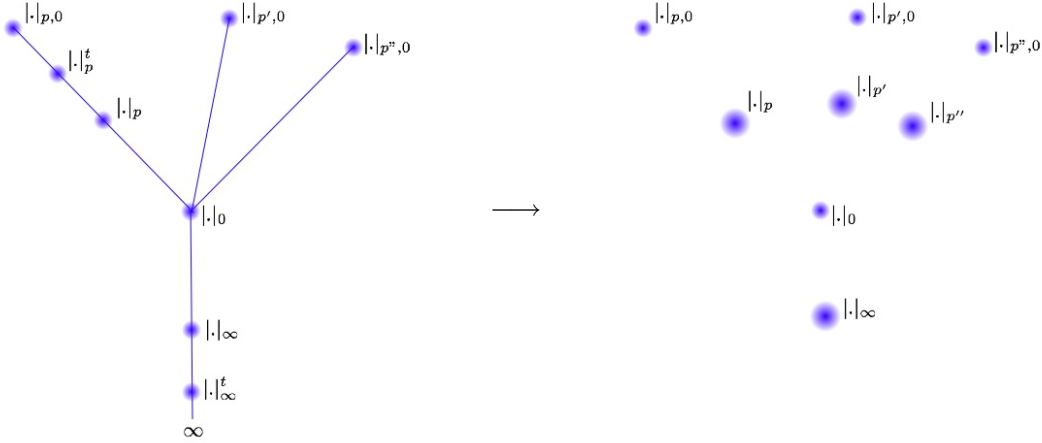


Figure 1: From Berkovich to Harmonious spectrum.

*Proof.* Let  $|\cdot| = |\cdot|(x) : \mathbb{Z} \rightarrow R$  be a tempered multiplicative seminorm that represents a multiplicative place  $x \in \text{Speh}^m(\mathbb{Z})$ . The kernel  $\mathfrak{p}$  of  $|\cdot|$  is a prime ideal of  $\mathbb{Z}$ . If  $\mathfrak{p} \neq 0$ , it is of the form  $\mathfrak{p} = (p)$  for a prime number  $p$  and  $|\cdot|$  factorizes through the finite field  $\mathbb{Z}/p\mathbb{Z}$ . The multiplicative seminorm  $|\cdot| : \mathbb{Z}/p\mathbb{Z} \rightarrow R$  is equivalent to the trivial seminorm  $|\cdot|_{p,0} : \mathbb{Z}/p\mathbb{Z} \rightarrow R_{\{1\}} = \{0, 1\}$  by Lemma 11. Now suppose that  $\mathfrak{p} = 0$  and  $|\cdot|$  is a generalized norm. If  $|2| \leq 1$ , then by Lemma 10,  $|\cdot|$  is non-archimedean. Lemma 12 shows that  $|\cdot|$  is equivalent to  $|\cdot|_p : \mathbb{Q} \rightarrow K_{p^{\mathbb{Z}}}$  or  $|\cdot|_0 : \mathbb{Q} \rightarrow K_{\{1\}}$ . If  $|2| > 1$ , then Theorem 1 shows that  $|\cdot|$  is multiplicatively equivalent to the usual archimedean seminorm  $|\cdot|_\infty : \mathbb{Z} \rightarrow \mathbb{Q}_+$ . All this shows that points of  $\text{Speh}^m(A)$  are exactly given by

$$\text{Speh}^m(A) = \{|\cdot|_p, |\cdot|_{p,0}, |\cdot|_\infty\}.$$

The multiplicative seminorm spectrum of  $\mathbb{Z}$  is

$$\mathcal{M}(\mathbb{Z}) = \{|\cdot|_p^t, |\cdot|_{p,0}, p \text{ prime}, t \in [0, \infty[ \} \cup \{|\cdot|_\infty^t, t \in [0, 1] \}.$$

as shown in Emil Artin's book (following Ostrowski) [Art67].  $\square$

### 3.4 The multiplicative harmonious affine line over $\mathbb{Z}$

We will now give a quite complete description of the points of the multiplicative harmonious affine line over  $\mathbb{Z}$ .

**Definition 19.** The harmonious affine line over  $\mathbb{Z}$ , denoted  $\mathbb{A}_{\mathbb{Z}}^{1,h}$  is  $\text{Speh}^m(\mathbb{Z}[X])$ .

Recall from Lemma 10 that the valuation spectrum  $\text{Spev}(\mathbb{Z}[X]) \subset \text{Speh}^m(\mathbb{Z}[X])$  is exactly the set of multiplicative seminorms such that  $|2| \leq 1$ .

**Lemma 13.** *Let  $x \in \text{Speh}^m(\mathbb{Z}[X])$  be a seminorm. If the image of  $x$  in  $\text{Speh}^m(\mathbb{Z})$  has a non-zero kernel  $(p)$ , then  $|\cdot(x)|$  is non-archimedean and there exists an irreducible polynomial  $P \in \mathbb{F}_p[X]$  such that*

1. *either  $|\cdot(x)|$  is equivalent to the trivial seminorm  $|\cdot|_{0,P,p} : \mathbb{F}_p[X]/(P) \rightarrow R_{\{1\}} = \{0, 1\}$ ,*
2. *or  $|\cdot(x)|$  is equivalent to the  $P$ -adic seminorm  $|\cdot|_P : \mathbb{F}_p[X] \rightarrow R_{P^{\mathbb{Z}}}$  given by  $|Q|_P = P^{-\text{ord}_P(Q)}$ .*

*Proof.* Since  $|p(x)| < 1$  and  $|\cdot(x)|$  is pre-archimedean, we get by Lemma 9 that it is non-archimedean. We already know that  $p\mathbb{Z}[X]$  is included in the kernel of  $|\cdot(x)|$  so that we have a factorization  $|\cdot(x)| : \mathbb{F}_p[X] \rightarrow R$ . If this factorization has a non-trivial kernel, then this kernel is a prime ideal of  $\mathbb{F}_p[X]$  generated by an irreducible polynomial  $P$ , and by Lemma 11,  $|\cdot(x)|$  is equivalent to  $|\cdot|_{0,P,p} : \mathbb{F}_p[X]/(P) \rightarrow R_{\{1\}} = \{0, 1\}$ . If the factorization  $|\cdot(x)| : \mathbb{F}_p[X] \rightarrow R$  has non-trivial kernel, then the subset  $\mathfrak{p} = \{P \in \mathbb{F}_p[X] \mid |P(x)| < 1\}$  is a prime ideal generated by an irreducible polynomial  $P$ . We prove similarly as in Lemma 12 that  $|\cdot|$  is equivalent to the multiplicative seminorm  $|\cdot|_P : \mathbb{F}_p[X] \rightarrow R_{P^{\mathbb{Z}}}$  given by  $|Q|_P = P^{-\text{ord}_P(Q)}$ .  $\square$

We recall for reader's convenience Huber and Knebusch's description of the (non-archimedean) valuation spectrum of a polynomial algebra in [HK94] on an algebraically closed field  $K$ , Proposition 3.3.2, in terms of ultrafilters of discs in  $K$ . Remark that  $\text{Spev}(\mathbb{Q}[T])$  is the quotient of  $\text{Spev}(\mathbb{Q}[T])$  by the Galois action.

Let  $K$  be an algebraically closed field.

**Definition 20.** Let  $|\cdot| : K \rightarrow R$  be a non-trivial non-archimedean valuation. A disc of  $K$  for  $|\cdot|$  is a subset  $S \subset K$  of the form

$$S = B^+(a, \gamma) = \{x \in K \mid |x - a| \leq |\gamma|\} \text{ or } S = B^-(a, \gamma) = \{x \in K \mid |x - a| < |\gamma|\}$$

for  $a, \gamma \in K$ . To a disc of  $K$  naturally corresponds a unique subset  $\tilde{S}$  of  $\text{Specv}(K[T])$  given by

$$\tilde{S} = \{x \in \text{Specv}(A) \mid |(X - a)(x)| \leq |\gamma(x)|\} \text{ or } \tilde{S} = \{x \in K \mid |(X - a)(x)| < |\gamma(x)|\}.$$

The set of discs is denoted  $C$ .

**Proposition 4** (Huber-Knebusch). *Let  $x \in \text{Specv}(K[X])$  be a valuation whose restriction  $|\cdot|$  to  $\mathbb{Q}$  is non-trivial. There exists a unique filter of discs  $F$  of  $K$  such that  $x \in \tilde{S}$  for all  $S \in F$ . Distinguishing four cases, we can give a precise description of  $x$ .*

1. *If there exists  $a \in K^*$  such that  $F = \{S \in C \mid a \in S\}$ , then  $x$  is given by*

$$P \mapsto |P(a)|.$$

2. *If there exists  $a \in K^*$ ,  $\gamma \in K^*$  with  $|\gamma| > 0$  and  $F = \{S \in C \mid B^+(a, \gamma) \subset S\}$ , then*

$$\left| \sum_{i=0}^n a_i (X - a)^i \right| (x) = \max(|a_i| \cdot |\gamma|^i \mid i = 0, \dots, n),$$

*i.e.,  $|\cdot|(x)$  is a generalized Gauss valuation.*

3. *If  $\cap_{S \in F} S = \emptyset$  then  $x$  is an immediate extension of  $|\cdot|$  to  $K(X)$  and can be constructed as follows. Let  $p(X)/q(X) \in K(X)^*$  be given. Choose  $S \in F$  which is disjoint to the zero set of  $p(X) \cdot q(X)$ . Then there exists  $\gamma \in K^*$  with  $|p(x)/q(x)| = |\gamma|$  for all  $x \in S$ , and we have*

$$|p(X)/q(X)|(x) = |\gamma|.$$

4. *Assume that  $x$  is not of the previous types. Choose  $a \in \cap_{S \in F} S$  and denote  $M = \{|\gamma| \in |K^*| \mid B^-(a, \gamma) \in F\}$ . Then  $M \subset |K^*|$  is a major subset (i.e. if  $x \in M$ ,  $y \in |K^*|$  with  $x \leq y$  then  $y \in M$ )*

$$\left| \sum_{i=0}^n a_i (X - a)^i \right| (x) = \max(|a_i| q^i)$$

*where the value group is  $|K^*| \times q^{\mathbb{Z}}$  with the ordering extending the one of  $|K^*|$  and such that  $M = \{|\gamma| \in |K^*|, q = |X - a| < |\gamma|\}$ . More precisely, we have, depending on  $M$ , the three following possibilities:*

- (a) *If  $M = \emptyset$  then  $|\gamma| < q = |X - a|$  for all  $\gamma \in K^*$  and if  $a_n \neq 0$  then*

$$\left| \sum_{i=0}^n a_i (X - a)^i \right| (x) = |a_n X^n|(x) = |a_n| q^n.$$

- (b) *If  $M = |K^*|$  then  $|X - a| = q < |\gamma|$  for all  $\gamma \in K^*$  and if  $a_{i_0} \neq 0$  then*

$$\left| \sum_{i=i_0}^n a_i (X - a)^i \right| (x) = |a_{i_0}| q^{i_0}.$$

(c) If  $M = \{|\gamma| \in |K^*|, |b| < |\gamma|\}$  then

$$\left| \sum_{i=i_0}^n a_i \left( \frac{X-a}{b} \right)^i \right| (x) = \max(|a_i|q^i).$$

With help of our temperation condition, a similar classification result also holds for archimedean tempered power-multiplicative seminorms.

**Definition 21.** Let  $|\cdot| : \mathbb{Q}[X] \rightarrow R$  be a seminorm. We say that  $|\cdot|$  is upper (resp. lower) bounded if for all  $P \in \mathbb{Q}[X] - \{0\}$ , there exists  $\lambda_P \in \mathbb{Q}^*$  (resp.  $\mu_P \in \mathbb{Q}^*$ ) such that  $|P| \leq |\lambda_P|$  (resp.  $|\mu_P| \leq |P|$ ).

**Theorem 2.** Let  $x \in \text{Speh}^m(\bar{\mathbb{Q}}[X])$  be a tempered multiplicative seminorm such that  $|2| > 1$ . Distinguishing various cases, we can give the following description of  $x$ .

1. If  $|\cdot|(x)$  has non-trivial kernel then there exists  $a \in \bar{\mathbb{Q}}$  such that  $|\cdot|(x)$  is multiplicatively equivalent to

$$|\cdot|(a)|_{\mathbb{C}} : \bar{\mathbb{Q}}[X] \rightarrow \mathbb{R}_+,$$

where  $|\cdot|_{\mathbb{C}}$  is the usual complex norm composed with an embedding  $\bar{\mathbb{Q}} \rightarrow \mathbb{C}$ .

2. Otherwise, if  $|\cdot|(x)$  is upper and lower bounded then it is multiplicatively equivalent to an  $\mathbb{R}_+$ -valued seminorm and it extends to  $\mathbb{C}[X]$ . More precisely, there exists a point  $a$  of  $\mathbb{C}$  (not equal to a point of  $\bar{\mathbb{Q}}$ ) such that  $|\cdot|(x)$  is equivalent to  $|\cdot|(a)|_{\mathbb{C}}$ .
3. If  $|\cdot|(x)$  is upper but not lower bounded then there exists  $a \in \bar{\mathbb{Q}}$  such that  $|\cdot|(x)$  is multiplicatively equivalent to  $|\cdot|_{2,a}$ , where

$$\left| \sum_{i=i_0}^n a_i (X-a)^i \right|_{2,a} = \max(|a_i|_{\mathbb{C}} q^i) = |a_{i_0}|_{\mathbb{C}} q^{i_0}$$

for  $a_{i_0} \neq 0$ , where the value halo is  $\mathbb{R}_{+[\times]} R_{q^{\mathbb{Z}}}$  and  $q < r$  for all  $r \in \mathbb{R}_+$ .

4. If  $|\cdot|(x)$  is lower but not upper bounded then  $|\cdot|(x)$  is multiplicatively equivalent to  $|\cdot|_{2,\infty}$ , where

$$\left| \sum_{i=0}^n a_i (X-a)^i \right|_{2,\infty} = \max(|a_i|_{\mathbb{C}} q^i) = |a_n|_{\mathbb{C}} q^n$$

for  $a_n \neq 0$ , where the value halo is  $\mathbb{R}_{+[\times]} R_{q^{\mathbb{Z}}}$  and  $q > r$  for all  $r \in \mathbb{R}_+$ .

*Proof.* First recall from Proposition 3 that since  $|2(x)| > 1$ ,  $|\cdot|(x)|_{\mathbb{Z}}$  is multiplicative and equivalent to the usual archimedean norm  $|\cdot|_{\mathbb{C}}$ . Since  $|\cdot|(x)$  is multiplicative, its kernel is a prime ideal of  $\bar{\mathbb{Q}}[X]$ . If this ideal is non-trivial, it is of the form  $(X-a)$  for  $a \in \bar{\mathbb{Q}}$ . Then  $|\cdot|(x)$  factors through  $\bar{\mathbb{Q}}[X]/(X-a) \cong \bar{\mathbb{Q}}$  and it is equivalent to  $|\cdot|(a)|_{\mathbb{C}}$ . From now on, we suppose that  $|\cdot|$  has trivial kernel.

Now suppose that  $|\cdot|(x)$  is both upper and lower bounded. Then the natural inclusion  $i : \bar{\mathbb{Q}} \rightarrow \bar{\mathbb{Q}}[X]$  is continuous for the topology induced on both rings by  $|\cdot|(x)$ . Indeed, if



$|b(x)| \neq 0$  with  $b \in \bar{\mathbb{Q}}[X]$  then there exists  $b' = \mu_b \in \bar{\mathbb{Q}}$  such that  $|b'(x)| \leq |b(x)|$  so that  $B(0, |b'|) \subset B(0, |b|)$  and  $i$  is continuous. This implies that  $i$  extends to  $i : \mathbb{C} \rightarrow \mathbb{C}[X]$ . We put on  $R$  the uniform structure generated by the sets

$$U_{|\lambda|} = \{(x, y) \in R \times R \mid \max(x, y) \leq \min(y + |\lambda|, x + |\lambda|)\}$$

for  $\lambda \in \mathbb{Q}^*$ , and denote by  $\hat{R}$  the completion of  $R$  for this uniform structure. Then  $|\cdot| : \bar{\mathbb{Q}}[X] \rightarrow R$  is uniformly continuous and extends at least to  $|\cdot| : \mathbb{C}[X] \rightarrow \hat{R}$ . We will now replace  $R$  by  $\hat{R}$ . The fact that  $|\cdot|$  is multiplicatively equivalent to an  $\mathbb{R}_+$ -valued power-multiplicative seminorm follows from forthcoming Lemma 14. It is well known that such a seminorm is the composition of the complex norm with evaluation at a point  $a$  of  $\mathbb{C}$ .

If we suppose that  $|\cdot|(x)$  is upper but not lower bounded, since  $|\cdot|$  is multiplicative, the non-archimedean seminorm  $|\cdot|' : \bar{\mathbb{Q}}[X] \rightarrow K_{R^*/R_2}$  (where  $R_2$  is the convex subgroup generated by  $|2|$ ) is multiplicative, so that  $\mathfrak{p} = \{x \in \bar{\mathbb{Q}}[X] \mid |x|' < 1\}$  is a prime ideal in  $\bar{\mathbb{Q}}[X]$ . If it is reduced to  $(0)$  then  $|\cdot|$  is also lower bounded which is a contradiction. We thus have  $\mathfrak{p} = (X - a)$  for some  $a \in \bar{\mathbb{Q}}$ . Let  $P$  be a non-zero polynomial. We want to prove that for all  $x \in \bar{\mathbb{Q}}$ ,  $x \neq a$ ,  $|X - x| = |x - a|$ . First remark that  $\frac{|X-x|}{|x-a|} \leq 1 + \frac{|X-a|}{|x-a|}$  so that

$$\frac{|X - x|^n}{|x - a|^n} \leq \sum_{p=0}^n \binom{n}{p} \cdot \frac{|(X - a)|^p}{|x - a|^p}.$$

But we know that  $|X - a|$  is smaller than any  $|\lambda|$  for  $\lambda \in \bar{\mathbb{Q}}^*$  so that  $\binom{n}{p} \cdot \frac{|X-a|^p}{|x-a|^p} \leq 1$ . This gives us  $\frac{|X-x|^n}{|x-a|^n} \leq n + 1$  so that  $|X - x| \leq |x - a|$ . Since  $(X - x) \notin \mathfrak{p}$ , we know that  $|X - x| \geq |\mu_x| > 0$  for some  $\mu_x \in \bar{\mathbb{Q}}^*$ . This allows us to prove as before that  $|X - x| \geq |x - a|$  so that we have equality. By induction, if  $P = (X - a)^n \cdot u \prod_x (X - x)$  with  $x \neq a$  and  $u \in \bar{\mathbb{Q}}^*$ , then  $|P| = |(X - a)^n \cdot u \prod_x (x - a)| = |X - a|^n \cdot |u \prod_x (x - a)|$ . Since  $|\cdot|_{\bar{\mathbb{Q}}}$  is equivalent to  $|\cdot|_{\mathbb{C}}$  and  $|X - a|$  is smaller than any  $|\lambda|$  for  $\lambda \in \bar{\mathbb{Q}}^*$ , we get that  $|\cdot|$  is multiplicatively equivalent to  $|\cdot|_{2,a}$  for  $a \in \bar{\mathbb{Q}}$ .

Now suppose that  $|\cdot|$  is lower but not upper bounded. We will show that if  $P = \sum_{i=0}^m a_i X^i$  with  $a_m \neq 0$  then  $|P| = |a_m X^m|$ . Since  $|\cdot|$  is multiplicative, the non-archimedean seminorm  $|\cdot|' : \bar{\mathbb{Q}}[X] \rightarrow K_{\mathbb{R}_+^*/R_2}$  is multiplicative and extends to  $\bar{\mathbb{Q}}(X)$ . The subset  $A = \{R \in \bar{\mathbb{Q}}[X] \mid |R|' \leq 1\}$  is a valuation ring in  $\bar{\mathbb{Q}}(X)$  and  $\mathfrak{p} = \{R \in \bar{\mathbb{Q}}(X) \mid |R|' < 1\}$  is a prime ideal in  $A$ . Remark that since  $|\cdot|$  is not upper bounded, we know from forthcoming Lemma 15 that for all  $x \in \bar{\mathbb{Q}}$  and all  $\lambda \in \mathbb{Q}^*$ ,  $|X - x| > |\lambda|$ . This implies that  $A$  is a subring of  $\bar{\mathbb{Q}}(X)$  that contains the algebra generated by  $\{\frac{1}{X-x}\}_{x \in \bar{\mathbb{Q}}}$ . Remark that using the decomposition of rational fractions in simple parts, we know that a quotient  $\frac{P}{Q}$  of two polynomials  $P$  and  $Q$  is in  $A$  if and only if  $\deg(P) \leq \deg(Q)$ . Indeed, if  $\frac{P}{Q} \in A$ , we can write

$$\frac{P}{Q} = R_0 + \sum_{i=1}^m \frac{c_i}{Q_i}$$

with  $c_i \in \bar{\mathbb{Q}}^*$ ,  $Q_i \in \bar{\mathbb{Q}}[X]$  of non-zero degree and  $R_0 \in \bar{\mathbb{Q}}[X]$  of degree equal to  $\deg(P) - \deg(Q)$  (saying that  $\deg(0) = -\infty$ ). If  $\deg(P) > \deg(Q)$  then  $\deg(R_0) > 0$  so that

$|R_0| > |\lambda|$  for all  $\lambda \in \bar{\mathbb{Q}}^*$ . But since the  $Q_i$ 's are of non-zero degree, we have  $\left|\frac{c_i}{Q_i}\right|' \leq 1$ . Now remark that

$$|R_0|' = \left| \frac{P}{Q} - \sum_{i=1}^m \frac{c_i}{Q_i} \right|' \leq \max \left( \left| \frac{P}{Q} \right|', \left| \frac{c_i}{Q_i} \right|' \right).$$

This implies that  $|R_0|' \leq 1$  so that  $|R_0|$  is bounded by some  $|\lambda|$  for  $\lambda \in \bar{\mathbb{Q}}^*$ , which contradicts the hypothesis. We thus have shown that

$$A = \left\{ \frac{P}{Q} \in \bar{\mathbb{Q}}(X) \mid \deg(P) \leq \deg(Q) \right\}.$$

Now denote  $P = \sum_{i=0}^m a_i X^i$  with  $a_m \neq 0$  and  $S = P - a_m X^m = \sum_{i=0}^{m-1} a_i X^i$ . Consider the inequality

$$\left( \frac{|P|}{|a_m X^m|} \right)^r \leq \left( 1 + \frac{|S|}{|a_m X^m|} \right)^r \leq \sum_{j=0}^r \binom{r}{j} \left| \frac{|S|}{|a_m X^m|} \right|^j.$$

Since  $\deg(S) < m = \deg(a_m X^m)$ , we have  $|S|' < |a_m X^m|'$  which means that for all  $\lambda \in \mathbb{Q}^*$ ,  $|S| < |a_m X^m| \cdot |\lambda|$ . This implies in particular that  $\binom{r}{j} \left| \frac{|S|}{|a_m X^m|} \right|^j \leq 1$  so that

$$\left( \frac{|P|}{|a_m X^m|} \right)^r \leq r + 1$$

for all  $r \in \mathbb{N}$  and since  $R$  has tempered growth, we get  $|P| \leq |a_m X^m|$ . Similarly, using that  $a_m X^m = P - S$ , we obtain the inequality

$$\left( \frac{|a_m X^m|}{|P|} \right)^r \leq \left( 1 + \frac{|S|}{|P|} \right)^r \leq \sum_{j=0}^r \binom{r}{j} \left| \frac{|S|}{|P|} \right|^j,$$

and since  $\deg(S) < \deg(P)$ , we get that  $|P| \geq |a_m X^m|$  so that

$$\left| \sum_{i=0}^m a_i X^i \right| = |a_m X^m|.$$

This shows that  $|\cdot|$  is multiplicatively equivalent to  $|\cdot|_{2,\infty}$ . □

**Lemma 14.** *Let  $A$  be a  $\mathbb{C}$ -algebra and  $|\cdot| : A \rightarrow \mathbb{R}$  be a tempered power-multiplicative seminorm on  $A$  whose restriction to  $\mathbb{C}$  is multiplicatively equivalent to the usual archimedean norm  $|\cdot|_{\mathbb{C}}$ . If  $|\cdot|$  has trivial kernel and  $|\cdot|$  is both upper and lower bounded, then  $|\cdot|$  is equivalent to an  $\mathbb{R}_+$ -valued seminorm.*

*Proof.* Suppose that  $\text{Ker}(|\cdot|) = (0)$  and  $|\cdot|$  is both upper and lower bounded. Let  $P \in A$  be a non-zero element. Then there exist  $\lambda_P, \mu_P \in \mathbb{C}^*$  such that  $0 < |\lambda_P| \leq |P| \leq |\mu_P|$ . Let  $\mu_\infty \in \mathbb{C}^*$  be such that  $|\mu_\infty|_{\mathbb{C}} = \sup\{|\mu_P|_{\mathbb{C}}, |\mu_P| \leq |P|\}$ . By construction, if  $|\lambda_P| \geq |P|$ ,  $|\mu_\infty| \leq |\lambda_P|$ . Now let  $\lambda_\infty \in \mathbb{C}^*$  be such that  $|\lambda_\infty|_{\mathbb{C}} = \inf\{|\lambda_P|_{\mathbb{C}}, |\lambda_P| \geq |P|\}$ . By

construction, if  $|\mu_P| \leq |P|$ ,  $|\lambda_\infty| \geq |\mu_P|$ . So we have  $|\mu_\infty| \leq |\lambda_\infty|$ . If  $|\mu_\infty| < |\lambda_\infty|$ , then there exists  $x \in \mathbb{C}^*$  such that  $|\mu_\infty| < |x| < |\lambda_\infty|$ . If  $|P| < |x|$ , then  $|x| \leq |\lambda_\infty|$  which is a contradiction. Similarly, if  $|P| > |x|$ , we get a contradiction. If  $|P| = |x|$ , we also get a contradiction. This shows that  $|\mu_\infty| = |\lambda_\infty|$ . By definition of these two, we also get  $|P| = |\mu_\infty| = |\lambda_\infty|$ , so that there exists  $\lambda_P \in \mathbb{C}^*$  such that  $|P| = |\lambda_P|$ . To sum up, for all  $P \in A$ , there exist  $\lambda \in \mathbb{C}^*$  such that  $|P| = |\lambda_P|$ , with  $|\lambda_P|_\infty = \inf\{|\lambda|_\infty, |P| \leq |\lambda_\infty|\} = \sup\{|\mu|_\infty, |P| \leq |\mu_\infty|\}$ . We define  $|P|_1 := |\lambda_P|_\infty$ . Since  $|\cdot|_{\mathbb{C}} \sim |\cdot|_{\mathbb{C}}$ , this is well defined. Moreover, it is a seminorm. Recall that we have  $|P + Q| \leq |2| \cdot \max(|P|, |Q|)$  so that  $|\lambda_{P+Q}| \leq |2| \cdot \max(|\lambda_P|, |\lambda_Q|) = \max(|2\lambda_P|, |2\lambda_Q|)$  because  $|\cdot|$  is multiplicative on  $\mathbb{C}$ . But this implies  $|P + Q|_1 \leq 2 \max(|P|_1, |Q|_1)$ . The fact that one can deduce from this inequality that  $|P + Q|_1 \leq |P|_1 + |Q|_1$  was known to E. Artin and can be found in [Art67], Theorem 3. We have  $|PQ| \leq |P| \cdot |Q|$  so that  $|\lambda_{PQ}| \leq |\lambda_P| \cdot |\lambda_Q|$  and since  $|\cdot|$  is multiplicative on  $\mathbb{C}$ , this implies  $|\lambda_{PQ}| \leq |\lambda_P \lambda_Q|$  and  $|PQ|_1 \leq |P|_1 \cdot |Q|_1$ . If  $|P| \cdot |Q| \leq |PQ|$  then  $|\lambda_P| \cdot |\lambda_Q| \leq |\lambda_{PQ}|$  and since  $|\cdot|$  is multiplicative on  $\mathbb{C}$ , this implies  $|\lambda_P \lambda_Q| \leq |\lambda_{PQ}|$  so that

$$|P|_1 \cdot |Q|_1 = |\lambda_P|_{\mathbb{C}} \cdot |\lambda_Q|_{\mathbb{C}} = |\lambda_P \cdot \lambda_Q|_{\mathbb{C}} \leq |\lambda_{PQ}|_{\mathbb{C}} = |PQ|_1.$$

This shows that  $|\cdot|$  and  $|\cdot|_1$  are multiplicatively equivalent and we conclude that  $|\cdot|_1 : \mathbb{C}[X] \rightarrow \mathbb{R}_+$  is a power-multiplicative seminorm.  $\square$

**Lemma 15.** *A power-multiplicative seminorm on  $\mathbb{C}[X]$  whose restriction to  $\mathbb{C}$  is equivalent to  $|\cdot|_{\mathbb{C}}$  is upper bounded if and only if there exist  $a \in \mathbb{C}$  and  $n \in \mathbb{Z}$  such that  $|X - a| \leq |n|$ .*

*Proof.* One of the implications is clear. If there exist  $a \in \mathbb{C}$  and  $n_0 \in \mathbb{Z}$  such that  $|X - a| \leq |n_0|$ , then for all  $x \in \mathbb{C}$ , there exists  $n_x \in \mathbb{Z}$  such that we have

$$|X - x| \leq |2| \max(|X - a|, |x - a|) \leq \max(|2| \cdot |n_0|, |2| \cdot |n_x|) = \max(|2 \cdot n_0|, |2 \cdot n_x|)$$

so that there exists  $m_x \in \mathbb{Z}$  such that  $|X - x| \leq |m_x|$ . If  $P$  is a polynomial, let  $P = u \times \prod_{x_i} (X - x_i)$  be the decomposition of  $P$  in prime factors. Then we have  $|P| = |u| \cdot \prod_{x_i} |X - x_i|$  and we get an  $n_P \in \mathbb{Z}$  such that  $|P| \leq |n_P|$ .  $\square$

## 4 Analytic spaces

We now want to define analytic functions on some convenient subspaces of harmonious spectra. We first have to define the value ring for analytic functions at a given place. This will be given by what we call the multiplicative completion.

### 4.1 Ball topologies and multiplicative completions

**Definition 22.** Let  $A$  be ring and  $|\cdot| : A \rightarrow R$  be a seminorm. An open ball in  $A$  for  $|\cdot|$  is a subset of the form  $B(x, |a|) := \{z \in A \mid |z - x| < |a|\}$  for  $a \in A$  such that  $|a| > 0$ . The ball neighborhood topology on  $A$ , denoted  $\tau_{|\cdot|}^{bn}$  if it exists, is the topology for which the non-empty open balls form fundamental systems of neighborhoods of the points in  $A$ .

We now give the coarsest conditions on a seminorm  $|\cdot| : A \rightarrow R$  under which its ball neighborhood topology is well defined and induces a topological ring structure on  $A$ .

**Definition 23.** Let  $A$  be a ring and  $|\cdot| : A \rightarrow R$  be a seminorm. We say that  $|\cdot|$  has tiny balls if

1. for all  $|a| > 0$ , there exists  $|a'| > 0$  such that

$$B(0, |a'|) + B(0, |a'|) \subset B(0, |a|);$$

2. for all  $|a| > 0$  and  $x \in A$ , there exists  $|c| > 0$  such that

$$x.B(0, |c|) \subset B(0, |a|);$$

3. for all  $|a| > 0$ , there exists  $|a'| > 0$  such that

$$B(0, |a'|).B(0, |a'|) \subset B(0, |a|).$$

4. for all  $|a| > 0$ , there exists  $|a'| > 0$  such that  $-B(0, |a'|) \subset B(0, |a|)$ .

*Remark 12.* Since  $R$  is totally ordered, the third condition is automatic. Indeed, if  $|a| > 1$ , then there exists  $a' = 1$  such that  $B(0, |1|).B(0, |1|) \subset B(0, |a|)$ . Suppose that  $|a| = 1$ . If there exists no  $c$  in  $A$  such that  $|c| < 1$ , then  $B(0, |1|) = \{0\} = B(0, |1|).B(0, |1|)$ . If there exists  $c \in A$  such that  $|c| < 1 = |a|$ , then  $|c|^2 < |c|$  and  $B(0, |c|).B(0, |c|) \subset B(0, |c|) \subset B(0, |a|)$ . If  $|a| < 1$  then  $|a|^2 < |a|$  and  $B(0, |a|).B(0, |a|) \subset B(0, |a|)$ .

*Remark 13.* If  $|\cdot|$  is square-multiplicative, the fourth condition is also automatic because  $|-a| = |a|$  for all  $a \in A$  so that  $-B(0, |a|) = B(0, |a|)$ .

**Proposition 5.** Let  $A$  be a ring and  $|\cdot| : A \rightarrow R$  be a seminorm on  $A$  that has tiny balls. Then the ball neighborhood topology for  $|\cdot|$  exists and it induces a ring topology on  $A$ .

*Proof.* Consider the set  $\mathcal{B} := \{B(0, |a|) | 0 < |a| \in |A|\}$  of parts of  $A$ . This set is a filter basis since

1. the intersection  $B(0, |a|) \cap B(0, |a'|)$  is equal to  $B(0, \min(|a|, |a'|))$ ;
2. since  $R$  is positive, it is not empty because  $B(0, 1) \in \mathcal{B}$ , and  $\emptyset \notin \mathcal{B}$  because  $0 \in B(0, |a|)$  for all  $|a| > 0$ .

Now by hypothesis, this filter basis is such that

1. for all  $B(0, |a|) \in \mathcal{B}$ , there exists  $B(0, |a'|) \in \mathcal{B}$  such that  $B(0, |a'|) + B(0, |a'|) \subset B(0, |a|)$ ;
2. for all  $B(0, |a|) \in \mathcal{B}$ , there exists  $B(0, |a'|) \in \mathcal{B}$  such that  $-B(0, |a'|) \subset B(0, |a|) \in \mathcal{B}$ .

Then by Bourbaki [Bou71], Chap. 3, §1.2, Proposition 1, there exists only one topology on  $A$  compatible with its addition and such that  $\mathcal{B}$  is the filter basis of the filter of neighborhoods of the unit 0 of  $(A, +)$ . This topology is the ball neighborhood topology. Remark now that by hypothesis,

1. for all  $x \in A$  and  $B(0, |a|) \in \mathcal{B}$ , there exists  $B(0, |c|) \in \mathcal{B}$  such that  $x.B(0, |c|) \subset B(0, |a|)$ .
2. for all  $B(0, |a|) \in \mathcal{B}$ , there exists  $B(0, |a'|) \in \mathcal{B}$  such that  $B(0, |a'|).B(0, |a'|) \subset B(0, |a|)$ .

Then by Bourbaki [Bou71], Chap. 3, §6.3, the topological structure defined above is a topological ring structure on  $A$ .  $\square$

**Definition 24.** Let  $|\cdot| : A \rightarrow R$  be a seminorm on a ring  $A$ . If  $|\cdot|$  has tiny balls, the completion of the separated quotient  $A/\overline{\{0\}}$  of  $A$  for its topological ring structure is called the separated completion of  $A$  for  $|\cdot|$  and denoted  $\hat{A}_{|\cdot|}$ . Let  $A_M$  be the localization of  $A$  with respect to the multiplicative subset  $M = M_{|\cdot|}$  of  $|\cdot|$ . If  $|\cdot| : A_M \rightarrow R$  has tiny balls, we say that  $|\cdot|$  has multiplicatively tiny balls and call the corresponding completion  $\mathcal{A}(|\cdot|)$  of  $A_M$  the multiplicative completion of  $|\cdot|$ .

**Proposition 6.** Let  $|\cdot| : K \rightarrow R$  be a multiplicative seminorm on a field  $K$ . If

- either there exists  $u \in K$  such that  $|u| > 2$ ,
- or there exists  $|a| > 0$  such that  $|b| < |a|$  implies  $|b| = 0$ ,

then  $K$  has tiny balls for  $|\cdot|$  so that  $|\cdot|$  induces a ring topology on  $K$ . This topology is separated.

*Proof.* If there exists  $|a| > 0$  such that  $|b| < |a|$  implies  $|b| = 0$ , then  $B(0, |a|) = \{0\}$  (because the kernel of  $|\cdot|$  is an ideal that must be reduced to  $\{0\}$  since  $K$  is a field) and all conditions for  $K$  to have tiny balls are clearly fulfilled. Moreover, the corresponding topology is the discrete topology on  $K$  and is separated. Suppose now that for all  $|a| > 0$ , there exists  $|a'|$  such that  $0 < |a'| < |a|$ . Let  $b, c \in K$  be such that  $|b| > 0$  and  $|c| > 0$ . Applying the hypothesis to  $|b/a| = |b|/|a|$ , we show that there exists  $d \in K$  such that  $|d| < |b/a|$ , i.e.  $0 < |d|.|a| < |b|$ . This implies the second condition for  $|\cdot|$  to have tiny balls. Now remark that by hypothesis, there exists  $u \in K$  such that  $|u| > 2$ . Let  $a' \in K$  be such that  $0 < |a'| < |a|$ . Then if we let  $d = a/u$ , we have  $|d| + |d| \leq \frac{|a'|}{2} + \frac{|a'|}{2} = |a'| < |a|$ . This is the second condition for  $|\cdot|$  to have tiny balls. By Remarks 12 and 13, since  $|\cdot|$  is multiplicative, we have proved that  $|\cdot|$  has tiny balls and this implies that  $|\cdot|$  induces a ring topology on  $A$ . Since  $|\cdot|$  is multiplicative, its kernel is trivial. Suppose that  $x$  and  $y$  are two distinct elements of  $K$ . Then  $|x - y| > 0$  in  $R$  and we know by what we did above that there exists  $|d| > 0$  such that  $0 < |d| + |d| < |x - y|$ . Now if  $x \in B(x, |d|) \cap B(y, |d|)$ , then

$$|x - y| \leq |x - z| + |y - z| \leq |u| + |u| < |x - y|,$$

which is a contradiction, so that  $B(x, |d|) \cap B(y, |d|) = \emptyset$  and the topology is separated.  $\square$

**Corollary 4.** Let  $|\cdot| : K \rightarrow R$  be a tempered multiplicative seminorm on a field  $K$ . Then  $K$  has tiny balls for  $|\cdot|$ .

*Proof.* From Proposition 6, we are reduced to suppose that there exists no  $|a| > 0$  such that  $|b| < |a|$  implies  $|b| = 0$ . We will show that there exists  $u \in K$  such that  $|u| > 2$ . Indeed, if  $|x| \leq 2$  for all  $x \in K$ , then  $|x|^n \leq 2$  for all  $n \in \mathbb{N}$  and  $x \in K$  but since  $R$  has tempered growth, this implies  $|x| \leq 1$  for all  $x \in K$ . Since there exists  $x$  such that  $0 < |x| < 1$ , we have  $|1/x| = 1/|x| > 1$ , which gives a contradiction. So there exists a  $u \in K$  such that  $|u| > 2$ . Proposition 6 concludes the proof.  $\square$

The definition of the notion of multiplicative completion was given to get the following.

**Corollary 5.** *Let  $|\cdot| : A \rightarrow R$  be a tempered multiplicative seminorm on a ring  $A$ . Then  $|\cdot|$  has multiplicatively tiny balls so that the multiplicative completion of  $A$  for  $|\cdot|$  is well defined and it is isomorphic to the completion of the residue field  $\text{Frac}(A/\mathfrak{p}_{|\cdot|})$  with respect to its induced seminorm.*

*Proof.* Since  $|\cdot|$  is multiplicative, its set of multiplicative elements is  $M_{|\cdot|} = A - \mathfrak{p}_{|\cdot|}$  where  $\mathfrak{p}_{|\cdot|}$  is the prime ideal that is the support of  $|\cdot|$ . The corresponding localization is the local ring  $A_{(\mathfrak{p})}$  and  $|\cdot|$  factorizes through it. By Proposition 6, the field  $A_{(\mathfrak{p})}/\mathfrak{p} = \text{Frac}(A/\mathfrak{p})$  is separated for the topology induced by  $|\cdot|$  and it is equal to the separated quotient of  $A_{(\mathfrak{p})}$ . We also know that the extension  $|\cdot| : \text{Frac}(A/\mathfrak{p}) \rightarrow R$  has tiny balls so that its completion is well defined. It is equal to the multiplicative completion of  $A$  for  $|\cdot|$ .  $\square$

## 4.2 The functoriality issue for multiplicative completions

Let  $f : B \rightarrow A$  be a ring morphism and  $|\cdot| : A \rightarrow R$  be a seminorm. We will now give conditions that ensure that  $f$  induces a morphism

$$f : \mathcal{A}(|f(\cdot)|) \rightarrow \mathcal{A}(|\cdot|)$$

between the multiplicative completions of  $|f(\cdot)| : B \rightarrow R$  and  $|\cdot| : A \rightarrow R$ .

Remark that the set of multiplicative elements (see Definition 5) for a seminorm is not functorial in ring morphisms meaning that if  $f : B \rightarrow A$  is a ring morphism and  $|\cdot| : A \rightarrow R$  is a seminorm, then we don't necessarily have  $f(M_{|f(\cdot)|}) \subset M_{|\cdot|}$ . This problem does not appear in the case of multiplicative seminorms but we have to solve it in the non-multiplicative case. This motivates the following definition.

**Definition 25.** Let  $|\cdot| : A \rightarrow R$  be a seminorm on a ring  $A$  and  $M_A \subset A$  be the set of multiplicative elements for  $|\cdot|$ . We say that  $|\cdot|$  has a functorial multiplicative subset if for all ring inclusion  $f : B \subset A$ , we have an inclusion of  $M_B \subset M_A$  of the corresponding sets of multiplicative elements for  $|\cdot|$ .

*Example 8.* A multiplicative seminorm  $|\cdot| : A \rightarrow R$  has clearly a functorial multiplicative subset given by the complement  $M_{|\cdot|} = \{a \in A \mid |a| \neq 0\}$  of the corresponding prime ideal  $\mathfrak{p}_{|\cdot|} = \{a \in A \mid |a| = 0\}$ .

*Remark 14.* Completions of rings with respect to seminorms are not functorial in ring homomorphisms. For example, if  $|\cdot|_{2,0} : \mathbb{Q}(T) \rightarrow R = \mathbb{R}_{+,trop}[\times]\mathbb{R}_+$  is given by

$$\left| \sum_{i=i_0}^n a_i T^i \right|_{2,0} = (e^{-i_0}, |a_{i_0}|_p)$$

for  $a_{i_0} \neq 0$ , then the natural morphism

$$\mathbb{Q} \rightarrow \mathbb{Q}(T) \rightarrow R$$

does not induce a topological ring homomorphism from  $\mathbb{Q}_p$  to the completion of  $\mathbb{Q}(T)$ . Indeed, the inverse image of  $B(0, |T|)$  in  $\mathbb{Q}$  by the inclusion  $\mathbb{Q} \rightarrow \mathbb{Q}(T)$  is  $\{0\}$  and it is not open in  $\mathbb{Q}$  for the  $p$ -adic topology. In fact, the completion of  $\mathbb{Q}(T)$  for  $|\cdot|_{2,0}$  only depends on the  $T$ -adic valuation  $|P|_0 = e^{-i_0}$  and it gives  $\mathbb{Q}((T))$ . We thus need a more general notion of “completion” that is functorial with respect to sequences

$$A \xrightarrow{f} B \xrightarrow{|\cdot|} R$$

with  $f : A \rightarrow B$  a ring homomorphism and  $|\cdot| : B \rightarrow R$  a reasonable seminorm on  $B$ .

*Remark 15.* Let’s translate the above remark in terms of numbers. The notion of local number (or element of a completion) is not functorial in ring morphisms. This could be repaired by replacing the usual completion of  $(A, |\cdot|)$  by (say) the subset

$$\mathbb{A}_{(A, |\cdot|)}^{1,h} = \{x \in \mathbb{A}_A^1 \mid |(x)|_A \sim |\cdot|\}$$

of the harmonious affine line, which is clearly functorial by definition. In this sense, an  $(A, |\cdot|)$ -completed number would just be a particular seminorm on  $A[X]$ . This is not that strange because any number  $a$  in  $\mathbb{R}$  or  $\mathbb{Q}_p$  can be seen as such a thing in  $\mathbb{A}_{(\mathbb{Z}, |\cdot|)}^{1,h}$  given by  $|(a)|_\infty$  and  $|(a)|_p$  respectively. However, this setting would prevent us from adding numbers because there is no way to properly add seminorms with values in different halos. Perhaps one could be inspired by Conway’s theory of surreal numbers [Con94] to deal with this problem. The functoriality issue of analytic functions is certainly at the heart of our difficulties, and this is not to be hidden. The idea of this remark will be explored further in Section 5.

**Definition 26.** Let  $A$  be a ring and  $|\cdot| : A \rightarrow R$  be a seminorm on  $R$ . We say that  $|\cdot|$  has a functorial multiplicative completion if for all ring homomorphism  $f : B \rightarrow A$ , if we denote  $M_B \subset B$  and  $M_A \subset A$  the corresponding multiplicative subsets, we have that

1.  $f(M_B) \subset M_A$  (i.e.  $|\cdot|$  has a functorial multiplicative subset),
2.  $|f(\cdot)| : B \rightarrow R$  has multiplicatively tiny balls and  $|f(\cdot)|$  extends to the completion of  $B_{M_B}$ ,
3. for all  $|a| > 0$  in  $|A_{M_A}|$ , there exists  $b$  in  $B_{M_B}$  such that  $|b| > 0$  and  $B(0, |b|) \subset f^{-1}(B(0, |a|))$ .

If  $|\cdot| : A \rightarrow R$  is multiplicative and has functorial multiplicative completion, we will say that it has functorial residue fields.

*Example 9.* A tempered multiplicative seminorm  $|\cdot| : A \rightarrow R$  has functorial residue fields if and only if for all ring homomorphism  $f : B \rightarrow A$ , the corresponding (injective) morphism

$$f : \text{Frac}(B/\mathfrak{p}_B) = K_B(|\cdot|) \rightarrow K_A(|\cdot|) = \text{Frac}(A/\mathfrak{p}_A)$$

between the residue fields is continuous, i.e., for all  $a \in K_A^*$ , there exists  $b \in K_B^*$  such that  $B(0, |b|) \subset f^{-1}(B(0, |a|))$ .

**Lemma 16.** *A multiplicative seminorm  $|\cdot| : A \rightarrow \mathbb{R}_+$  has functorial residue fields.*

*Proof.* We can reduce to the case of a field  $A$ . Let  $f : B \rightarrow A$  be a field morphism. Remark that given  $a \in A^*$ , the condition that there exists  $b \in B^*$  such that  $B(0, |b|) \subset f^{-1}(B(0, |a|))$  is clearly fulfilled if  $|a| \geq 1$ . Indeed, we then have  $B(0, |1|) \subset f^{-1}(B(0, |1|)) \subset f^{-1}(B(0, |a|))$ . We thus restrict to the case  $|a| < 1$ . If there exists  $b \in B$  such that  $0 < |b| < 1$ , then there exists  $n \in \mathbb{N}$  such that  $|b^n| = |b|^n < |a|$  this implies that  $B(0, |b^n|) \subset f^{-1}(B(0, |a|))$ . Otherwise, we have  $|B| = \{0, 1\}$  and  $B(0, |1|) = \{0\} \subset f^{-1}(B(0, |a|))$ .  $\square$

### 4.3 The analytic spectrum of a ring

**Definition 27.** Let  $A$  be a ring. The analytic spectrum of  $A$  is the subset  $\text{Speh}^a(A)$  of  $\text{Speh}^m(A)$  of the tempered multiplicative spectrum given by seminorms  $|\cdot|(x)$  that have a functorial multiplicative completion.

**Definition 28.** Let  $A$  be a ring and

$$R\left(\frac{a_1, \dots, a_n}{b}\right) = \{x \in \text{Speh}^a(A) \mid |a_i(x)| < |b(x)|, d|b \Rightarrow d \text{ is multiplicative for } |\cdot|(x)|\}$$

be an open rational subset of  $\text{Speh}^a(A)$ . Let  $\tilde{R} \subset \text{Speh}^{pasm}(A)$  be the subset composed of  $R$  and of the set of  $\mathbb{R}_+$ -valued square-multiplicative seminorms that have functorial multiplicative completion and are  $\mathbb{Z}$ -multiplicative. The ring of analytic series on  $\tilde{R}$  is the completion  $\mathcal{O}(\tilde{R})$  of  $A[b^{-1}]$  for the topology induced by the natural map

$$A[b^{-1}] \rightarrow \prod_{x \in \tilde{R}} \mathcal{A}(x).$$

Let  $U$  be an open subset of  $\text{Speh}^a(A)$ . A rule

$$f : U \rightarrow \prod_{x \in \text{Speh}^a(A)} \mathcal{A}(x)$$

such that  $f(x) \in \mathcal{A}(x)$  is called an analytic function if for all  $x_0 \in \text{Speh}^a(A)$ , there exists an open rational domain  $R = R\left(\frac{a_1, \dots, a_n}{b}\right)$  contained in  $U$  and containing  $x_0$  such that  $f|_R$  is in the image of  $\mathcal{O}(\tilde{R})$  by the natural map

$$\mathcal{O}(\tilde{R}) \rightarrow \prod_{x \in R} \mathcal{A}(x).$$

*Remark 16.* Another approach to the definition of analytic function (that would prevent us from using the halo of positive real numbers  $\mathbb{R}_+$  and follows the viewpoint of Remark 15) will be studied in Section 5.



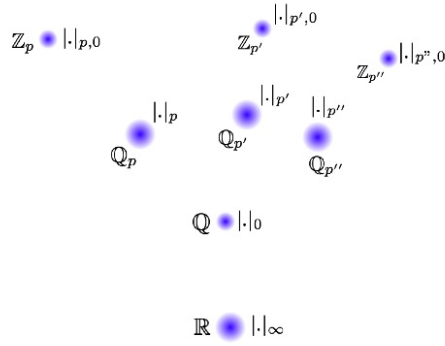


Figure 2: Germs of analytic functions on  $\text{Speh}^a(\mathbb{Z})$ .

#### 4.4 The analytic spectrum of $\mathbb{Z}$

**Proposition 7.** *The analytic and harmonious spectrum of  $\mathbb{Z}$  are identified. Germs of analytic functions on  $\text{Speh}^a(\mathbb{Z})$  can be described by figure 2 (see also figure 1).*

*More precisely, depending on the point  $x \in \text{Speh}^a(\mathbb{Z})$ , the corresponding residue field (resp. germs of analytic functions) are*

- $\mathbb{F}_p$  (resp.  $\mathbb{Z}_p$ ) if  $x = |\cdot|_{p,0}$ ,
- $\mathbb{Q}_p$  (resp.  $\mathbb{Q}_p$ ) if  $x = |\cdot|_p$ ,
- $\mathbb{R}$  (resp.  $\mathbb{R}$ ) if  $x = |\cdot|_\infty$ ,
- $\mathbb{Q}$  (resp.  $\mathbb{Q}$ ) if  $x = |\cdot|_0$ .

Moreover, for  $m \neq 0$ , sections on  $\{x | 0 < |m(x)|\}$  are identified with  $\mathbb{Z}[1/m]$  with its trivial topology. In particular, global sections are identified with  $\mathbb{Z}$ .

*Proof.* The statement on residue fields follows from the definition. If a rational domain  $D = \{|n_1| < |m|, \dots, |n_k| < |m|\}$  contains the trivial norm, then  $n_i = 0$  for all  $i = 1, \dots, k$  and  $\mathcal{O}^h(D)$  is the completion of  $\mathbb{Z}[1/m]$  with respect to the topology induced by the natural map

$$\mathbb{Z}[1/m] \rightarrow \mathbb{Q} \times \mathbb{R} \times \left( \prod_n \mathbb{Q}_n \right) \times \left( \prod_{p \nmid m} \mathbb{F}_p \right)$$

because the points of  $D$  are given by  $D = \{|\cdot|_0\} \cup \{|\cdot|_p, \forall p\} \cup \{|\cdot|_{p,0}, \forall p \nmid m\} \cup \{|\cdot|_\infty\}$  and the  $\mathbb{R}_+$ -power-multiplicative seminorms are of the form  $|\cdot|_n$  for  $n \in \mathbb{Z} - \{0, \pm 1\}$  and  $|\cdot|_{0,n}$  for  $n \nmid m$ . Since  $\mathbb{Q}$  above is equipped with the discrete topology (induced by the trivial valuation  $|\cdot|_0$ ), we have  $\hat{\mathcal{O}}(D) = \mathbb{Z}[1/m]$  equipped with the discrete topology. More precisely, germs of analytic functions at  $|\cdot|_0$  are equal to  $\varinjlim_m \mathbb{Z}[1/m] = \mathbb{Q}$ . Now suppose that  $D$  does not contain the trivial norm. Then it is a disjoint union of rational domains

of the form  $\{|p| < 1\}$  and  $\{0 < |p|, |p| < 1\}$ , so that we can suppose that  $D$  is of one of these two forms. In the first case, we have  $D = \{|\cdot|_p, |\cdot|_{p,0}\}$ , and  $\tilde{\mathcal{O}}(D)$  is the completion of  $\mathbb{Z}$  with respect to the natural map

$$\mathbb{Z} \rightarrow \mathbb{Q}_p \times \mathbb{F}_p,$$

i.e,  $\tilde{\mathcal{O}}(D) = \mathbb{Z}_p$ . More precisely, in this case,  $D$  is the smallest rational subset that contains  $|\cdot|_{p,0}$  so that germs of analytic functions around  $|\cdot|_{p,0}$  are also given by  $\mathbb{Z}_p$ . In the second case, we have  $D = \{|\cdot|_p\}$  and  $\tilde{\mathcal{O}}(D)$  is the completion of  $\mathbb{Z}[1/p]$  with respect to the natural map

$$\mathbb{Z}[1/p] \rightarrow \mathbb{Q}_p.$$

There is a natural map from the completion  $\mathbb{Z}_p$  of  $\mathbb{Z}$  with respect to  $\mathbb{Z} \rightarrow \mathbb{Q}_p$  to  $\tilde{\mathcal{O}}(D)$  and  $p$  is invertible in  $\tilde{\mathcal{O}}(D)$  so that the natural map  $\mathbb{Z}_p \rightarrow \tilde{\mathcal{O}}(D)$  factorizes through  $\mathbb{Q}_p = \mathbb{Z}_p[1/p]$ , so that  $\tilde{\mathcal{O}}(D) \cong \mathbb{Q}_p$  and these are also the germs of analytic functions at  $|\cdot|_p$ . If  $D = \{1 < |2|\} = \{|\cdot|_\infty\}$ ,  $\tilde{\mathcal{O}}(D)$  is the completion of  $\mathbb{Z}[1/2]$  with respect to the natural map

$$\mathbb{Z}[1/2] \rightarrow \mathbb{R},$$

so that  $\tilde{\mathcal{O}}(D) = \mathbb{R}$  and these are also the germs of analytic functions at  $|\cdot|_\infty$ .  $\square$

## 4.5 The analytic affine line over $\mathbb{Z}$

We first describe the set of points of the analytic line over  $\mathbb{Z}$ .

**Proposition 8.** *The points of  $\mathrm{Sphe}^m(\mathbb{Z}[T])$  that are in the analytic affine line  $\mathrm{Sphe}^a(\mathbb{Z}[T])$  are given by the following list:*

1. *The trivial seminorm.*
2.  *$\mathbb{F}_p$ -points: (non-archimedean) seminorms whose restriction to  $\mathbb{Z}$  have a non-trivial kernel  $(p)$ . These are described in Lemma 13.*
3.  *$\mathbb{Q}_p$ -points: (non-archimedean) seminorms whose restriction to  $\mathbb{Z}$  give the  $p$ -adic norm. These are described in Proposition 4 and the only ones that are non-analytic are those associated in 4 of loc. cit. to a major subset  $M \subset |\mathbb{Q}^*|$  with  $M = \emptyset$  or  $M = |\mathbb{Q}^*|$  (i.e. infinitesimal neighborhoods of  $p$ -adic  $\bar{\mathbb{Q}}$ -points and infinitesimal neighborhood of infinity). More precisely, they are in bijection with the points of Huber's adic affine line  $\mathbb{A}_{\mathbb{Q}_p}^{1,ad} = \mathbb{A}_{\mathbb{Q}_p}^1 \times_{\mathrm{Spec}(\mathbb{Q}_p)} \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ .*
4.  *$\mathbb{R}$ -points: (archimedean) seminorms whose restriction to  $\mathbb{Z}$  give the usual archimedean norm. These are described in Theorem 2 and the only ones that are non-analytic are those of the form  $|\cdot|_{2,a}$  for  $a \in \bar{\mathbb{Q}}^*$  or  $|\cdot|_{2,\infty}$  in the notations of loc. cit. (i.e. infinitesimal neighborhoods of archimedean  $\bar{\mathbb{Q}}$ -points and infinitesimal neighborhood of infinity). To be precise, archimedean seminorms are  $\mathbb{R}_+$ -valued seminorms.*

We now describe the relation of our structural sheaf with the usual sheaf of holomorphic functions on the complex plane.

**Proposition 9.** *Let  $\text{Speh}^{\text{arch}}(\mathbb{Z}[T])$  be the archimedean open subset of  $\text{Speh}^a(\mathbb{Z}[T])$ , defined by the condition  $|2(x)| > |1|$ . If  $R \subset \text{Speh}^{\text{arch}}(\mathbb{Z}[T])$  is a rational domain and  $R' \subset \mathbb{C}$  is the corresponding rational domain in the complex plane, the natural inclusion  $R' \rightarrow R$  induces an isomorphism*

$$\mathcal{O}(R) \xrightarrow{\sim} \mathcal{H}ol(R')$$

where  $\mathcal{H}ol(R')$  denotes usual holomorphic functions on  $R'$  and  $\mathcal{O}(R)$  denotes the sections of the structural analytic sheaf of  $\text{Speh}^a(\mathbb{Z}[T])$  on the open subset  $R$ .

*Proof.* Remark first that  $\text{Speh}^{\text{arch}}(\mathbb{Z}[T])$  is identified with  $\mathbb{C}/c$  where  $c$  denotes complex conjugation. Let  $\tilde{R}$  be the subset of  $\text{Speh}^{\text{pasm}}(\mathbb{Z}[T])$  associated to  $R$  as in Definition 28. This set is identified with the set of spectrally convex compact subsets of  $\mathbb{C}/c$ . The map  $R' \subset \tilde{R}$  is the inclusion of points in the set of compact subsets. A function in  $\mathcal{O}(\tilde{R})$  is locally a limit of a Cauchy sequence of rational functions with no poles for the topology of convergence on all compacts. Such a limit is holomorphic and is uniquely determined by its values on  $R' \subset R$ . Moreover, every holomorphic function on  $R'$  is a local uniform limit on all compacts of polynomials. This shows that  $\mathcal{O}(R) \xrightarrow{\sim} \mathcal{H}ol(R')$  is an isomorphism.  $\square$

There is also a nice relation between our structural sheaf and the sheaf of analytic functions on Berkovich's analytic space.

**Proposition 10.** *Let  $\text{Speh}^p(\mathbb{Z}[T])$  be the  $p$ -adic open subset of  $\text{Speh}^a(\mathbb{Z}[T])$ , defined by the conditions  $0 < |p(x)| < |1(x)|$  and  $\mathbb{A}_{\mathbb{Q}_p}^{1,\text{ber}}$  be Berkovich's affine line over  $\mathbb{Q}_p$ . The natural map  $f : \mathbb{A}_{\mathbb{Q}_p}^{1,\text{ber}} \rightarrow \text{Speh}^p(\mathbb{Z}[T])$  induces an isomorphism of sheaves  $f^*\mathcal{O} \rightarrow \mathcal{O}^{\text{ber}}$  between the inverse image of the sheaf of analytic functions and Berkovich's sheaf of analytic functions.*

*Proof.* By definition of an analytic point, every point of  $\text{Speh}^p(\mathbb{Z}[T])$  is in  $\text{Speh}^v(\mathbb{Z}[T])$  and extends to a point in the valuation spectrum  $\text{Speh}^v(\mathbb{Q}_p[T])$  of  $\mathbb{Q}_p[T]$ . We thus have a natural map

$$\text{Speh}^p(\mathbb{Z}[T]) \rightarrow \text{Speh}^v(\mathbb{Q}_p[T])$$

and the natural map  $\mathbb{A}_{\mathbb{Q}_p}^{1,\text{ber}} \rightarrow \text{Speh}^v(\mathbb{Q}_p[T])$  factors through its image. The points in  $\text{Speh}^p(\mathbb{Z}[T])$  that are not in the image of this map are associated to a major subset  $M \subset |\mathbb{Q}^*|$  as in Proposition 4 and are of the form

$$|\sum_i a_i(X - a)^i|(x) = \max(|a_i|q^i)$$

with  $q < |\gamma|$  for all  $|\gamma| \in M$  or  $|\gamma| < q$  for all  $|\gamma| \in |\mathbb{Q}^*| - M$ , and are generalization of the corresponding Gauss point

$$|\sum_i a_i(X - a)^i|(x) = \max(|a_i||\gamma_M|^i)$$

for a given  $\gamma_M \in \mathbb{Q}^*$  associated to  $M$ . This means that the corresponding topology on  $\mathbb{Q}_p(T)$  is coarser than the Gauss norm topology. The topology on rational functions on a given open rational domain  $R$  of  $\text{Speh}^p(\mathbb{Z}[T])$  thus only depends on the compact subsets of the corresponding rational domain  $R'$  of  $\mathbb{A}_{\mathbb{Q}_p}^{1,\text{ber}}$ . Making  $R'$  smaller, one can identify this topology with the topology of uniform convergence on  $R'$ .  $\square$

## 5 Another approach to analytic functions

The functoriality issue for residue fields alluded to in Section 4.2 is quite problematic. This is mainly due to the fact that the notion of completion we use is not really adapted to higher rank valuations. Our desire to use non- $\mathbb{R}_+$ -valued valuations in analytic geometry is mainly due to the fact that these cannot be studied by model theoretic (i.e. algebraic, in some sense) methods, because the archimedean property of  $\mathbb{R}_+$  is not a first order logic property. The other reason is the natural appearance of higher rank valuations in the study of the  $G$ -topology on non-archimedean analytic spaces. We will now look for a different kind of completion, that is easily seen to be functorial, but have other drawbacks.

### 5.1 Definition of functorial generalized completions

Let  $(A, |\cdot|)$  be a seminormed ring. If  $(A, |\cdot|) = (\mathbb{Z}, |\cdot|_\infty)$  we would like our definition of completion to give back  $\mathbb{R}$ . We can start by saying that the completion is simply the Berkovich affine line  $\mathbb{A}_{(A, |\cdot|)}^{1, ber}$ , i.e., the set of multiplicative seminorms  $|\cdot| : A[X] \rightarrow \mathbb{R}_+$  whose restriction to  $A$  is the given seminorm. In the above case, we get  $\mathbb{C}/c$  where  $c$  denotes complex conjugation.

**Definition 29.** The pseudo-archimedean square multiplicative affine line on a seminormed ring  $(A, |\cdot|)$  is the space

$$\mathbb{A}^{1, pasm}(A, |\cdot|) := \text{Speh}_{|\cdot|}^{pasm}(A[X])$$

of pre-archimedean square-multiplicative seminorms  $|\cdot|'$  on  $A[X]$  such that  $|\cdot|'_A$  is multiplicatively equivalent to  $|\cdot|$ .

From now on, we allow ourselves to use various topologies (typically given by rational subsets with closed or open inequalities) on the spectra  $\text{Speh}^{pasm}$ . It seems that the closed inequalities topology is better adapted to non-archimedean spaces.

Recall that there is a natural map

$$A \rightarrow \mathbb{A}^{1, pasm}(A, |\cdot|)$$

given by  $a \mapsto [P \mapsto |P(a)|]$ . This induces a topology on  $A$ , and we can think of the topological space  $\mathbb{A}^{1, pasm}(A, |\cdot|)$  as a kind of generalized completion of  $A$  for this topology. Moreover, if  $f : (A, |\cdot|_A) \rightarrow (B, |\cdot|_B)$  is a ring morphism such that  $|\cdot|_B \circ f : A \rightarrow \mathbb{R}$  is (multiplicatively) equivalent to  $|\cdot|_A$ , then  $f$  extends to a continuous map

$$\hat{f} : \mathbb{A}^{1, pasm}(A, |\cdot|_A) \rightarrow \mathbb{A}^{1, pasm}(B, |\cdot|_B).$$

Let (SNRINGS) denote the category of seminormed rings with morphisms respecting the multiplicative equivalence class of seminorms. The functor

$$\mathbb{A}^{1, pasm} : (\text{SNRINGS}) \rightarrow (\text{TOP})$$

can be seen as a functorial generalized completion functor. For more flexibility, we will extend this a bit.

**Definition 30.** Let  $\text{Speh}^\bullet$  be a sub-quotient functor of  $\text{Speh}^{pasm} : (\text{RINGS}) \rightarrow (\text{TOP})$ . Let  $\mathbb{A}^{1,\bullet} : (\text{SNRINGS}) \rightarrow (\text{TOP})$  be the corresponding version of the affine line given by

$$\mathbb{A}^{1,\bullet}(A, |\cdot|) := \{x \in \text{Speh}^\bullet(A[X]) \mid |.(x)|_A \sim |\cdot|\}.$$

Suppose further that the image of  $A$  in  $\mathbb{A}^{1,pasm}(A, |\cdot|)$  goes functorially in  $\mathbb{A}^{1,\bullet}(A, |\cdot|)$ . The  $\bullet$ -completion of  $(A, |\cdot|)$  is the space  $\mathbb{A}^{1,\bullet}(A, |\cdot|)$ .

Let us consider the functor  $\text{Speh}^\bullet = \text{Speh}^m \subset \text{Speh}^{pasm}$ . The tempered multiplicative affine line  $\mathbb{A}^{1,m}(\mathbb{Z}, |\cdot|_\infty)$  contains all seminorms of the form  $P \mapsto |P(x)|_\infty$  for  $x \in \mathbb{R}$ . It also contains finer infinitesimal points. Moreover, the tempered multiplicative affine line  $\mathbb{A}^{1,m}(\mathbb{Z}, |\cdot|_p)$  is equal to the valuative affine line  $\mathbb{A}^{1,v}(\mathbb{Q}, |\cdot|_p)$  whose separated quotient is the Berkovich affine line  $\mathcal{M}(\mathbb{Q}[T], |\cdot|_p)$  of multiplicative seminorms  $|\cdot| : \mathbb{Q}[T] \rightarrow \mathbb{R}_+$  such that  $|\cdot|_{\mathbb{Q}} \sim |\cdot|_p$ . It contains  $\mathbb{Q}_p$  as a subset but is much bigger.

*Remark 17.* The main advantage of such a completion procedure is that it is functorial. Its main drawback is that it does not give rings but just topological spaces of local functions.

## 5.2 Definition of foanalytic functions

Let  $\text{Speh}^\bullet$  be a sub-quotient functor of  $\text{Speh}^{pasm}$  as in last Section. Let  $A$  be a ring and

$$U = R \left( \frac{a_1, \dots, a_n}{b} \right) = \{x \in \text{Speh}^\bullet(A) \mid |a_i(x)| \leq |b(x)| \neq 0, d|b \Rightarrow d \text{ is multiplicative for } |.(x)|\}$$

be a (closed-inequalities) rational domain in  $\text{Speh}^\bullet(A)$ . Let  $\mathbb{A}_U^{1,\bullet} \subset \text{Speh}^\bullet(A[X])$  be the set of seminorms  $|\cdot| : A[X] \rightarrow R$  such that  $|\cdot|_A$  is multiplicatively equivalent to an  $x \in U$ .

**Definition 31.** The space of foanalytic <sup>2</sup> functions on  $U \subset \text{Speh}^\bullet(A)$  is the space  $\mathcal{B}^\bullet(U)$  of continuous sections of the natural projection

$$\pi : \mathbb{A}_U^{1,\bullet} \rightarrow U.$$

The evaluation morphisms for polynomials at elements  $a \in A$  induces a natural map

$$\text{ev} : A \rightarrow \mathcal{B}^\bullet(U)$$

which factors in

$$A[1/b] \rightarrow \mathcal{B}^\bullet(U)$$

because  $b$  is multiplicative for every seminorm  $x$  in  $U$ . This shows that the Zariski pre-sheaf  $\mathcal{O}_{alg}$  on rational domains naturally maps to the pre-sheaf  $\mathcal{B}^\bullet$ .

The possibility of varying the sub-quotient functor  $\text{Speh}^\bullet$ , and eventually taking sub-functors of  $\mathcal{B}^\bullet$ , makes our theory quite flexible, but the comparison with usual global analytic spaces does not seem to be an easy task. We have however included these constructions in our work because they are the only way we found of having a functorial version of the completion procedure that seems necessary to define a flexible notion of analytic function.

---

<sup>2</sup>Foanalytic is a shortcut for functorial analytic.

## Acknowledgments

The author thanks P. Almeida, V. Berkovich, M. Bermúdez, A. Ducros, I. Fesenko, R. Huber, J. Poineau, A. Thuillier, B. Toen, G. Skandalis and M. Vaquié for useful discussions around the subject of global analytic geometry. He is particularly grateful to R. Huber who authorized him to reproduce his own ideas in subsection 3.2. He also thanks University of Paris 6 and Jussieu’s Mathematical Institute for financial support and excellent working conditions.

## References

- [Art67] Emil Artin. *Algebraic numbers and algebraic functions*. Gordon and Breach Science Publishers, New York, 1967.
- [Ber90] Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.
- [Bou64] N. Bourbaki. *Éléments de mathématique. Fasc. XXX. Algèbre commutative. Chapitre 5: Entiers. Chapitre 6: Valuations*. Actualités Scientifiques et Industrielles, No. 1308. Hermann, Paris, 1964.
- [Bou71] N. Bourbaki. *Éléments de mathématique. Topologie générale. Chapitres 1 à 4*. Hermann, Paris, 1971.
- [Con94] J. H. Conway. The surreals and the reals. In *Real numbers, generalizations of the reals, and theories of continua*, volume 242 of *Synthese Lib.*, pages 93–103. Kluwer Acad. Publ., Dordrecht, 1994.
- [Dur07] Nikolai Durov. *New Approach to Arakelov Geometry*. arXiv.org:0704.2030, 2007.
- [Gro60] A. Grothendieck. Éléments de géométrie algébrique. I. Le langage des schémas. *Inst. Hautes Études Sci. Publ. Math.*, (4):228, 1960.
- [HK94] Roland Huber and Manfred Knebusch. On valuation spectra. In *Recent advances in real algebraic geometry and quadratic forms (Berkeley, CA, 1990/1991; San Francisco, CA, 1991)*, volume 155 of *Contemp. Math.*, pages 167–206. Amer. Math. Soc., Providence, RI, 1994.
- [Hub93] R. Huber. Continuous valuations. *Math. Z.*, 212(3):455–477, 1993.
- [Poi07] Jérôme Poineau. *Espaces de Berkovich sur  $\mathbb{Z}$* . Thèse. université de Rennes 1, 2007.
- [Poi08] Jérôme Poineau. *La droite de berkovich sur  $\mathbb{Z}$* . 2008.
- [Pre98] Alexander Prestel. *Model theory for the real algebraic geometer*. Pisa , Istituti editoriali e poligrafici internazionale – c1998, Pisa, 1998.

- [Sch90] N. Schwartz. Compactification of varieties. *Ark. Mat.*, 28(2):333–370, 1990.
- [Sou92] C. Soulé. *Lectures on Arakelov geometry*, volume 33 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1992. With the collaboration of D. Abramovich, J.-F. Burnol and J. Kramer.
- [Tat67] J. T. Tate. Fourier analysis in number fields, and Hecke’s zeta-functions. In *Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965)*, pages 305–347. Thompson, Washington, D.C., 1967.
- [Tat71] John Tate. Rigid analytic spaces. *Invent. Math.*, 12:257–289, 1971.